AN APPLICATION OF STOCHASTIC FLOWS TO RIEMANNIAN FOLIATIONS

ALAN MASON

Abstract. A stochastic flow is constructed on a frame bundle adapted to a Riemannian foliation on a compact manifold. The generator A of the resulting transition semigroup is shown to preserve the basic functions and forms, and there is an essentially unique strictly positive smooth function ϕ satisfying $A^*\phi = 0$. This function is used to perturb the metric, and an application of the ergodic theorem shows that there exists a bundle-like metric for which the basic projection of the mean curvature is basic-harmonic.

1. INTRODUCTION

To set the stage for the present work, let us begin by recalling the classical construction of Eells and Elworthy (see, e.g., [Bi], [IW]). For M a compact manifold with Riemannian metric g and orthonormal frame bundle $\mathcal{O}(M)$, let $Y_i, i = 1, \cdots, n = \dim M$ be the canonical horizontal vector fields on $\mathcal{O}(M)$ for an affine connection ∇ that preserves the metric. The idea of Eells and Elworthy is to consider the stochastic differential equation $dR_t = \sum_{i=1}^n Y_i(R_t)dw_t^i$, $R(0) = r_0$, and the associated semigroup $(S_t f)(z) = \int_\Omega f(\pi(R(t, r, \omega)))\, P_0^W(d\omega)$. Here f is a continuous function defined on M; the w^i are the components of standard Brownian motion on $\mathbb{R}^n$; the differentials are taken in the Stratonovich sense; $z = \pi(r)$ is the projection of the initial frame r at which the flow starts; and P_0^W is Wiener measure on the space Ω of continuous paths starting at the origin. By using the globally defined vector fields Y_i on the frame bundle and then projecting, this construction gives useful information while getting around the fact that M is usually not parallelizable. Thus, $S_t f(z)$ does not depend on the choice of frame r over $z \in M$ and we have a well-defined object on M. Moreover, by Ito's formula and the Markov property of the flow, $S_t f$ satisfies the heat equation $\frac{\mathrm{d}}{\mathrm{dt}} S_t f(z) = AS_t f$ where the second-order elliptic operator A depends on the connection ∇.

More precisely, the following facts are known.

1) If f is a smooth function on M then $\widehat{A}(f \circ \pi) = (Af) \circ \pi$, where $\pi : \mathcal{O}(M) \to M$ is the canonical projection. Here $\widehat{A} = \frac{1}{2}\sum_{i=1}^{n} Y_i^2$, $A = \frac{1}{2}\Delta + b$, Δ is the Laplacian for the Levi-Cività connection, and b is the so-called drift field. Moreover, given any vector field b on M, there exists a metric-preserving affine connection ∇ such that b arises in this fashion.

2) There exists a strictly positive smooth function ϕ on M satisfying $A^* \phi = 0$, where A^* is the L^2 adjoint of A. Moreover, ϕ is unique up to multiplication by a constant (see Proposition 6.1 below).

3) The (nonsymmetric) heat kernel for A on functions is strictly positive and the ergodic theorem holds: $\lim_{t\to\infty} S_t f(z) = \int_M f d\mu$, where μ is the unique probability measure associated with ϕ.

The above results hold quite generally, but as they stand there is no contact with geometry. It seems resonable that more information of a purely geometric nature should be obtainable if the drift field b itself is geometrically well-motivated.

Pursuing this, let us suppose that we have some structure $\mathcal{S}$ on M, that is, a decomposition of M into smooth orbits of a group action, say, or as leaves of a foliation, and suppose that this lifts to a structure $\widetilde{\mathcal{S}}$ on $\mathcal{O}(M)$ in the sense that each member $\widetilde{\Xi}$ of $\widetilde{\mathcal{S}}$ projects under π to a member Ξ of $\mathcal{S}$. Finally, suppose that the metric-preserving connection ∇ used in the Eells–Elworthy construction has the following property: If r_0 and r_1 are two initial frames in $\widetilde{\Xi}_0$ then the corresponding flows $R(t, r_0, \omega)$ and $R(t, r_1, \omega)$ respect the structure after projection, i.e., for almost every ω and every $t \geqslant 0$, $\pi \circ R(t, r_0, \omega)$ and $\pi \circ R(t, r_1, \omega)$ belong to the same set Ξ_t. Then the semigroup S_t will also preserve the structure; for instance, if the function f is constant on each set in the structure, the same will be true of $S_t f$.

Any connection ∇ with the above property will be of interest because the Eells–Elworthy construction then preserves useful geometric data. In particular, the associated drift field b will be a fundamental geometric object. In the present paper we show that the above ideas can be fully implemented for Riemannian foliations, which seem almost tailor-made for our purposes. Although the geometry has some novel features, the probabilistic techniques employed are standard.

We can now outline our results. The key fact about Riemannian foliations that we need is the existence of bundle-like metrics. These are used to lift the structure $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ on $\mathcal{O}(M)$. Lemma 2.1, a standard result for Riemannian submersions, depends essentially on (3), while Lemma 2.2 uses nothing more than the characteristic property (2) of bundle-like metrics. The connection $\nabla^{\oplus} = P\nabla P + P^{\perp}\nabla P^{\perp}$ is chosen for the Eells–Elworthy construction; that this is the right choice is shown

in Lemmas 3.1 and 3.3. The former lets us reduce to the adapted frame bundle ${}^{\mathcal{F}}\mathcal{O}(M)$, while the latter, our main technical result, shows that the transverse (deterministic) flows respect the foliation structure. This is not true for unrestricted flows, because Lemma 2.1 is valid only for tangent vectors X perpendicular to the leaves. This means that the generator of the transverse transition semigroup arising from the construction will not be elliptic, but for the moment this causes no problems.

We next pass to the transverse stochastic flow in the standard way, and Lemma 3.4 shows that the associated semigroup preserves the basic functions. We write T_t for the transverse semigroup, reserving S_t for the full semigroup. Lemma 3.5 is a general result showing equality of transverse semigroups acting on basic functions under changes of metric. All our results for T_t are seen to hold already at the level of individual trajectories. This is true in particular of Theorem 5.4, which is therefore merely a translation into heat-equation terms of the geometry of $\nabla^{\oplus}$ using the Eells–Elworthy machinery.

Things become a little more interesting when we restrict our attention to functions and examine what facts 2) and 3) above have to say in our situation. Here the need for ellipticity leads us to consider the full semigroup S_t. Lemma 5.2 shows that $S_t f = T_t f$ for all basic functions f, even though the full flow does not respect the foliation structure; thus $S_t f = T_t f$ for basic f does not follow by taking limits from a corresponding result that holds at the level of individual trajectories. The proof of Lemma 5.2 uses the uniqueness of solutions of the heat equation and also, in an essential way, the fact that we have reduced to the subbundle ${}^{\mathcal{F}}\mathcal{O}(M)$ and the fact that the transverse semigroup T_t preserves the basic functions (Lemma 3.4).

Lemma 5.1 establishes that the drift b corresponding to $\nabla^{\oplus}$ is just $\kappa/2$, where κ is the mean curvature field; as expected, this is a fundamental geometric object. We are now in a position to bring fact 3) to bear, leading to Theorem 6.2. Remark 1 reflects the abundance of bundle-like metrics; for the purposes of Theorem 6.2, it would be sufficient just to dilate by ϕ or ϕ_{b}. Section 7 considers an example in some detail. We remark here that since one cannot hope to actually calculate ϕ or its basic component ϕ_{b} explicitly, an essential role is played by the ergodic theorem as the only tool available for getting a handle on the behavior of ϕ_{b} when the bundle-like metric is varied.

The author believes that an explicit geometric-probabilistic approach is the most natural, if not only, way to study the kinds of questions considered here in their full generality. However, if one is just interested in Theorem 6.2, the question

arises of whether the probability theory can be eliminated. We will discuss this further at the end of the paper.

This work differs significantly from and largely supersedes the author's thesis [Ma], to which we can nonetheless refer for a few omitted proofs.

2. THE ADAPTED FRAME BUNDLE AND ITS FOLIATION

Let M be a compact manifold of dimension n equipped with a foliation $\mathcal{F}$ of dimension p. There is an atlas of simple charts (U_α, ϕ_α) on M of the form

$$\phi_\alpha : \ U_\alpha \approx \mathbb{R}^p \times \mathbb{R}^q$$

with distinguished coordinates

$$\{z_j\} = \{x_i, y_{a-p}\}, \quad i = 1, \cdots, p, \ a = p+1, \cdots, n,$$

where the x_i are along the foliation $\mathcal{F}$ and the y_{a-p} are transverse to it. Each subset $y = $ const of U is called a plaque and is contained in a leaf of $\mathcal{F}$; $q =$ codim $\mathcal{F}$.

Let $q : z = (x, y) \mapsto \overline{z} := y$ also denote the quotient map (defined locally on each chart), with differential

$$q_* : T_zM \to \overline{Q}_z \equiv T_zM/T_z\mathcal{F}, \qquad X \mapsto \overline{X}.$$

Given a Riemannian metric g on M, we obtain a splitting

$$TM = T\mathcal{F} \oplus Q \approx T\mathcal{F} \oplus \overline{Q}$$

of the exact sequence of bundles

$$0 \to T\mathcal{F} \to TM \to \overline{Q} \to 0$$

where $Q = (T\mathcal{F})^\perp$, the orthogonal complement of $T\mathcal{F}$ with respect to g. If (U', ϕ') is another simple chart in the atlas for $(M, \mathcal{F})$, then the transition map $\phi' \circ \phi^{-1}$ on $U \cap U'$ is of the form

$$(x, y) \mapsto (x'(x, y), y'(y)), \tag{1}$$

i.e., plaques go to plaques.

We recall that a Riemannian foliation is one for which there exists an atlas satisfying the following condition: the Jacobians ${\phi' \circ \phi^{-1}}_*$ define maps $U \cap U' \to O(q)$, where $O(q)$ is the group of orthogonal matrices acting on $\mathbb{R}^q$. Equivalently, we can regard $\mathbb{R}^q$ as a local model space equipped with a Riemannian metric g_T which is preserved by the transition maps. In general g_T will not coincide with the standard Euclidean metric on $\mathbb{R}^q$ and may have curvature; we will therefore write $\overline{M/\mathcal{F}}$ rather than $\mathbb{R}^q$ for the local model space. A transverse covariant

derivative ∇^T on $\overline{M/\mathcal{F}}$ is uniquely determined by g_T in the usual way by the Koszul formula. We will deal only with Riemannian foliations.

Definition 1. A vector field $\xi(z) = \sum \xi_j(z)\frac{\partial}{\partial z_j}$ is said to be **foliate** (or **projectable**) if it projects locally via q to a vector field on the local model space $\overline{M/\mathcal{F}}$, that is, if the functions $\xi_j(z)$ for $j = p+1, \cdots, n$ depend only on the y coordinate in $z = (x, y)$.

Definition 2. A form $\theta \in A^r(M)$ is said to be **basic** if for every $X \in T\mathcal{F}$ we have

$$i_X(\theta) = 0 \text{ and } i_X(d\theta) = 0,$$

where i_X denotes contraction with X. Thus θ is basic if and only if it involves only the transverse coordinates y: $\theta = \sum_K \theta_K(z)dz^K$ in terms of distinguished local coordinates $z = (x, y)$, where $K = (k_1, \cdots, k_r)$ is an increasing multi-index with $k_1 > p$, and the coefficients θ_K depend only on y. In particular, a function is basic if and only if it is constant along leaves. We denote the spaces of basic functions and forms by $C_{\mathrm{b}}(M)$ and $\mathcal{A}_{\mathrm{b}}(M)$, respectively. The Riemannian metric g defines an L^2-projection P_{b} onto the subcomplex of basic forms and gives a decomposition $\theta = \theta_{\mathrm{b}} + \theta_{\mathrm{o}}$ into basic and basic-orthogonal components.

Definition 3. The Riemannian metric g on M is **bundle-like** if and only if $\mathcal{L}_Z g = 0$ whenever $Z \in T\mathcal{F}$ is along the leaves; here $\mathcal{L}_Z$ denotes Lie derivative. In other words,

$$\begin{aligned} &\text{for any two local vector fields } X, Y \in (T\mathcal{F})^{\perp}, \text{ the function} \\ &z \mapsto g_z(X, Y) \text{ is constant along the leaves wherever } X \text{ and } Y \text{ are foliate.} \end{aligned} \tag{2}$$

We will consider only bundle-like metrics g that are compatible with the given transverse metric g_T in the following sense:

$$g_z(e, f) = (g_T)_{\overline{z}}(\overline{e}, \overline{f}) \;\; \forall\, e, f \in T_z\mathcal{F}^{\perp}. \tag{3}$$

This is meaningful because the transverse metric g_T is preserved under the coordinate transformations in the defining atlas. Such metrics can be constructed as follows. Given any Riemannian metric g' on M, let $V \subset TM$ be the distribution defining the foliation $\mathcal{F}$, and let P be the g'-orthogonal projection on V. Set $g(X, Y) = g'(PX, PY) + g_T(\overline{X}, \overline{Y})$ [Mo, Prop. 3.3].

There is an orthogonal splitting

$$TM = T\mathcal{F} \oplus T\mathcal{F}^{\perp}$$

into vertical and horizontal subspaces. We write $P, P^{\perp}$ for the orthogonal projections on $T\mathcal{F}$ and $(T\mathcal{F})^{\perp}$, respectively. Because g is compatible with g_T (3), in each chart U_i with $q : U_i \to \overline{M/\mathcal{F}}, \quad z \mapsto \overline{z} = y$ is a Riemannian submersion onto the model quotient space, i.e., the local quotient map q gives an isometry $T_z\mathcal{F}^{\perp} \approx T_{\overline{z}}\overline{M/\mathcal{F}}$.

Passing to forms, we have a splitting $T^*M = T^*\mathcal{F} \oplus Q^*$ into components along and transverse to the leaves. This induces a decomposition of the r-forms on M :

$$A^r(M) = \bigoplus_{u+v=r} A^u(Q) \otimes A^v(\mathcal{F}). \tag{4}$$

There is a corresponding filtration, with forms in $A^{u,v} = A^u(Q) \otimes A^v(\mathcal{F})$ said to be of type (u, v). With respect to this filtration, the exterior derivative decomposes as $d = d_{1,0} + d_{0,1} + d_{2,-1}$.

Let $\mathcal{O}(M) \xrightarrow{\pi} M$ be the principal bundle of orthonormal frames, and let ${}^{\mathcal{F}}\mathcal{O}(M)$ be the subbundle of frames $r = [z, (e_1, \cdots, e_p, e_{p+1}, \cdots, e_n)], \quad z \in M$, adapted to $\mathcal{F}$. That is, the first p vectors e_i are along the leaves, while the last q are in $T\mathcal{F}^{\perp}$.

In general, we say that a field of frames r (i.e., a local section of the bundle $\mathcal{GL}(M)$ of all frames, or a subbundle of it) is **foliate** if each element e_j is given by a foliate vector field near z. Expressing each e_j as a column vector in terms of the $\frac{\partial}{\partial z_k}$, we see that a frame in ${}^{\mathcal{F}}\mathcal{O}(M)$ has the form

$$r = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}. \tag{5}$$

The j-th frame element is

$$e_j = \sum_{k=1}^{n} e_j^k \, \partial/\partial z_k, \tag{6}$$

where k labels the row and j labels the column.

Because the metric g is bundle-like the Gram–Schmidt procedure, applied to a preferred basis

$$\partial/\partial z_1, \cdots, \partial/\partial z_p, \partial/\partial z_{p+1}, \cdots, \partial/\partial z_n$$

in a simple chart, yields foliate frames, i.e., the elements e_j $(1 \leqslant j \leqslant n)$ are foliate. Gram–Schmidt thus creates a foliate local orthonormal field of frames from a local chart.

The following result will be needed in the construction of the flow. We omit the straightforward proof, which uses (3) and the Koszul formula for ∇ and ∇^T.

Lemma 2.1. *If $X \in T_z\mathcal{F}^\perp$ then*

$$\overline{(P^\perp \nabla_X P^\perp \partial/\partial z_l)_z} = \left(\nabla^T_{\overline{X}} \overline{\frac{\partial}{\partial z_l}}\right)_{\overline{z}}.$$

As the bundle-like metric g varies, so do the spaces ${}^{\mathcal{F}}\mathcal{O}(M)$. We will regard them as lying in $\mathcal{GL}(M)$.

The adapted frame bundle ${}^{\mathcal{F}}\mathcal{O}(M) \xrightarrow{\pi} M$ has a natural foliation $\widetilde{\mathcal{F}}$, again of dimension p, which explicitly reflects the variation of the metric g along the leaves of $\mathcal{F}$. The leaves of $\widetilde{\mathcal{F}}$ are of the form

$$\widetilde{\mathcal{L}} = \{r' = [z = (x,y), \vec{e'}] \mid z \in \mathcal{L}, r' = \mathbf{gs}(r_0)\},$$

where $\mathcal{L}$ is a leaf of $\mathcal{F}$ and $r_0 = [z_0 = (x_0, y_0); \vec{e}]$ is some reference frame based at a point $z_0 \in \mathcal{L}$. The components of $r' = \mathbf{gs}(r_0) = [z, \vec{e'}],\ z \in \mathcal{L}$, are by definition given by

$$\begin{aligned}
e'_1 &= \frac{e_1}{\|e_1\|_{g_z}} \\
e'_2 &= \frac{e_2 - g_z(e_2, e'_1)e'_1}{\|e_2 - g_z(e_2, e'_1)e'_1\|_{g_z}} \\
&\ \vdots \\
e'_{p+1} &= \frac{e_{p+1} - \sum_{j=1}^p g_z(e_{p+1}, e'_j)e'_j}{\|e_{p+1} - \sum_{j=1}^p g_z(e_{p+1}, e'_j)e'_j\|_{g_z}} \\
&\ \vdots
\end{aligned} \tag{7}$$

Here the reference frame r_0 is extended in the obvious way to be a constant field in $\mathcal{GL}(M)$ in a simple chart about z_0: $r_0(z) = [z; \vec{e}]$, so that $e_j = e_j^k(z_0)\partial_k$ is a constant vector field. To make sense of this definition of $\widetilde{\mathcal{F}}$, we start with the fact that the Gram–Schmidt map $\mathbf{gs}$ is transitive: For z, z', z'' three points in a simple chart U, let $r' = \mathbf{gs}(r; z \to z'),\ r'' = \mathbf{gs}(r'; z' \to z''),\ \widehat{r} = \mathbf{gs}(r; z \to z'')$; then $\widehat{r} = r''$. This leads to a global equivalence relation: $r \sim r'$ if and only if r and r' both lie over the same leaf $\mathcal{L}$ and there exist a chain of overlapping charts U_i and frames $r_i \in {}^{\mathcal{F}}\mathcal{O}(M), z_i \equiv \pi(r_i) \in \mathcal{L} \cap U_i, 0 \leqslant i \leqslant N$, with $r = r_0, r' = r_N,\ z_i \in U_i \cap U_{i-1}$ for $1 \leqslant i \leqslant N$, and $r_{i+1} = \mathbf{gs}(r_i, z_i \to z_{i+1})$ for all i. This equivalence class of frames comprises the lifted leaf $\widetilde{\mathcal{L}}$ and defines the lifted foliation $\widetilde{\mathcal{F}}$. The

transitivity of Gram–Schmidt ensures that there is no dependence on the choice of reference frame $r_0 \in \widetilde{\mathcal{L}}$. It is easy to check that $\widetilde{\mathcal{F}}$ is a foliation, and for each leaf $\widetilde{\mathcal{L}}$, $\pi : \widetilde{\mathcal{L}} \to \mathcal{L}$ is a covering map.

Lemma 2.2. *The C coordinates are constant along a leaf $\widetilde{\mathcal{L}}$.*

Proof. Since the C coordinates of the first p vectors are identically zero for all frames r in ${}^{\mathcal{F}}\mathcal{O}(M)$, we start by considering e'_{p+1} in (7). Because g is bundle-like and the local vector field $z \mapsto e_{p+1} - \sum_{j=1}^{p} g_z(e_{p+1}, e'_j)e'_j$ is foliate and orthogonal to $T\mathcal{F}$, we have

$$\|e_{p+1} - \sum_{j=1}^{p} g_z(e_{p+1}, e'_j)e'_j\|_{g_z} = \|e_{p+1} - \sum_{j=1}^{p} g_{z_0}(e_{p+1}, e_j)e_j\|_{g_{z_0}} = 1.$$

The assertion of the Lemma is now clear for $e'_{p+1} = e_{p+1} - \sum_{j=1}^{p} g_z(e_{p+1}, e'_j)e'_j$. Consider next the numerator $e_{p+2} - \sum_{j=1}^{p+1} g_z(e_{p+2}, e'_j)e'_j$ of e'_{p+2}. By (2), we have

$$\begin{aligned} g_z(e_{p+2}, e'_{p+1}) &= g_z\left(e_{p+2} - \sum_{k=1}^{p} g_z(e_{p+2}, e'_k)e'_k, \quad e'_{p+1}\right) \\ &= g_{z_0}(e_{p+2}, e_{p+1}) = 0. \end{aligned}$$

Thus $\|e_{p+2} - \sum_{j=1}^{p+1} g_z(e_{p+2}, e'_j)e'_j\| \equiv 1$ by the same argument used for e'_{p+1}, and hence $e'_{p+2} = e_{p+2} - \sum_{j=1}^{p} g_z(e_{p+2}, e'_j)e'_j$. Thus, ${e'}^k_{p+2} = e^k_{p+2}$ for all $k > p$. Continuing in this way, we obtain ${e'}^k_a = e^k_a$ for all $a, k > p$. □

Since the leaf $\widetilde{\mathcal{L}}$ is not globally contained in a simple chart, we need to be more precise about the global meaning of Lemma 2.2. To this end, let C' be the corresponding coordinates in an overlapping chart U'; they are related to the coordinates C by the Jacobian $\overline{J}(x, y)$ of the transformation (1), which is independent of the coordinates x along the leaf $\mathcal{L}$, given by $y =$ const. Since the leaf $\widetilde{\mathcal{L}}$ lies over $\mathcal{L}$, we see that the C' are constant along $\widetilde{\mathcal{L}}$ and given by $C' = \overline{J}(x, y) \cdot C$, for any value of x corresponding to $z = (x, y), y =$ const, in the overlap $U \cap U'$. Given two frames $r_0, r_1 \in \widetilde{\mathcal{L}}$, we can join them by a path γ in $\widetilde{\mathcal{L}}$ and choose intermediate points $\rho_0 = r_0, \cdots, \rho_N = r_1$ on γ such that the portion of γ from ρ_i to ρ_{i+1} is contained in a simple chart U_i, and ρ_i, ρ_{i+1} belong to the same plaque in U_i. By following along these plaques, we see how the C coordinates for r_0 are related to those for r_1 (in general, there will of course be a dependence on the homotopy class of the path γ).

On the other hand, by (7) the frame coordinates in A transform by an invertible matrix in $GL(p)$. The condition that the frames be orthonormal at each point z implies in particular:

$$g_z(A, B+C) = 0, \qquad \text{or} \qquad g_z(A,B) = -g_z(A,C)$$

(in a convenient short-hand notation). Thus B is uniquely determined by $C, \mathcal{F}$, and the metric g_z; it does not depend on A, whose vectors merely span $T\mathcal{F}$. As we move along a leaf $\widetilde{\mathcal{L}}$, the metric varies and the B components adjust themselves so as to preserve orthogonality to $T\mathcal{F}$, the C components remaining constant by Lemma 2.2.

The structure group for ${}^{\mathcal{F}}\mathcal{O}(M)$ is $G = O(p)\times O(q) \subset O(n)$. A frame $r = [z; \vec{e}]$ at $z \in M$ can be regarded as a map

$$\mathbb{R}^p \times \mathbb{R}^q \to T_z M,\ (u,v) \mapsto \sum_1^p u_i e_i + \sum_{\alpha=p+1}^{n} v_{\alpha-p} e_\alpha.$$

The action of $\gamma = \gamma' \times \gamma''$ is given by

$$(r\cdot\gamma)(u,v) = \sum_1^p (\gamma'\cdot u)_i e_i + \sum_{\alpha=p+1}^{n} (\gamma''\cdot v)_{\alpha-p} e_\alpha,$$

where $(\gamma'\cdot u)_i = \sum_1^p (\gamma')_{ij} u_j$ and so on. Thus, the j-th frame element of $r\cdot\gamma$ is given by

$$(r\cdot\gamma)_j = \sum_i \gamma_{ij} e_i. \tag{8}$$

For $z_1, z_2 \in M$ and $r_1, r_2 \in {}^{\mathcal{F}}\mathcal{O}(M)$, we will write

$$z_1 \sim z_2, \quad r_1 \sim r_2, \quad \text{and } r_1 \sim r_2 \bmod O(p), \tag{9}$$

respectively, to mean that z_1 and z_2 lie on the same leaf $\mathcal{L}$ of $\mathcal{F}$; r_1 and r_2 lie on the same leaf $\widetilde{\mathcal{L}}$ of $\widetilde{\mathcal{F}}$; and $r_2 \in \widetilde{\mathcal{L}}\cdot\gamma$ for some $\gamma \in O(p)$, where $r_1 \in \widetilde{\mathcal{L}}$. Clearly, $r_1 \sim r_2 \bmod O(p)$ implies $\pi(r_1) \sim \pi(r_2)$.

Finally, for a given bundle-like metric g on M, we let ∇ denote the Levi-Cività connection on M and set

$$\nabla^{\oplus} = P\nabla P + P^{\perp}\nabla P^{\perp}.$$

Clearly, $\nabla^{\oplus}$ preserves the metric g since ∇ does.

3. CONSTRUCTION OF THE FLOW

To construct the flow we consider a simple chart U with coordinates $z = (x, y)$, in terms of which we have

$$\nabla^{\oplus}_{\partial_k} \partial_l = \sum_{i=1}^{n} {}^{\oplus}\Gamma^i_{kl} \partial_i,$$

where $\partial_i = \frac{\partial}{\partial z_i}$ and the ${}^{\oplus}\Gamma^i_{kl}$ are the Christoffel symbols. Suppose that $i > p$ and $l \leqslant p$. Then $\nabla^{\oplus}_{\partial_k} \partial_l = P\nabla^{\oplus}_{\partial_k} \partial_l \in T\mathcal{F}$, since $P^{\perp}\partial_l \equiv 0$. Hence

$$(10) \qquad {}^{\oplus}\Gamma^i_{kl} = 0 \quad \text{for } i > p, l \leqslant p.$$

Let $Y_a,\ 1 \leqslant a \leqslant n$, be the canonical horizontal vector fields on $\mathcal{GL}(M)$; they are uniquely determined by the two conditions

i) Y_a is horizontal for the connection $\nabla^{\oplus}$;

ii) $\pi_*(Y_a|_r) = r(E_a) \in T_z(M)$

for any frame $r \in \mathcal{GL}(M), \pi(r) = z$; here $E_a \in \mathbb{R}^n$ is the canonical unit vector and we regard r as a map $\mathbb{R}^n \to T_z(M)$. We note that because $\nabla^{\oplus}$ preserves the metric, the Y_a restrict to vector fields on the orthonormal frame bundle $\mathcal{O}(M)$.

In terms of local coordinates z, e^i_j on $\mathcal{GL}(M)$ the standard horizontal vector fields are given by [IW, Chap. V, Eq. (4.12)]

$$(11) \qquad Y_a = e^m_a \partial_m - {}^{\oplus}\Gamma^i_{kl}\, e^k_a e^l_j \partial/\partial e^i_j;$$

all indices range from 1 to n, the "vertical" coordinates e^i_j are given by $e_j = e^i_j\, \partial_i$, and repeated indices are summed.

We fix a vector field Y_a and consider the associated flow ${}_aR$ given by

$$(12) \qquad \begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} z^m(t) &= e^m_a(t) \\ \frac{\mathrm{d}}{\mathrm{dt}} e^i_j(t) &= -\sum_{k,l} {}^{\oplus}\Gamma^i_{kl}(z(t)) e^k_a(t) e^l_j(t) \end{aligned}$$

with initial condition ${}_aR(t = 0) = r_0$.

Definition 4. A flow $R(t, \cdot)$ will be said to be *adapted* to $\mathcal{F}$ if $\pi \circ R(t, r_0)$ respects $\mathcal{F}$ in the following sense:

$$\pi \circ R(t, r_0) \text{ varies in a leaf } \mathcal{L}_t \text{ as } r_0 \text{ varies in } \widetilde{\mathcal{L}}.$$

This condition is weaker than requiring that the flow be foliate for $\widetilde{\mathcal{F}}$. We will say that $R(t,\cdot)$ is *weakly adapted* to $\mathcal{F}$ if:

$$\text{for every basic } f \in C_{\mathrm{b}}(M), f(\pi(R(t,r_0))) \text{ is again basic,}$$

for any choice of initial frame r_0 over $z \in \mathcal{L}$. In other words, given $z \in M$, choose some frame $r_0 \in {}^{\mathcal{F}}\mathcal{O}(M)$ at z and let r_0' vary in the leaf $\widetilde{\mathcal{L}}$ containing r_0; then $f(\pi(R(t,r_0')))$ is constant.

In order for a flow $R(t,r_0)$ starting at $r_0 \in {}^{\mathcal{F}}\mathcal{O}(M)$ to be useful, it must preserve ${}^{\mathcal{F}}\mathcal{O}(M)$ and be adapted to $\mathcal{F}$. The next two lemmas will show that the flows ${}_aR$, $a = 1,\cdots,n$, have the necessary properties, even though they are not foliate for $\widetilde{\mathcal{F}}$.

Lemma 3.1. *Let the flows ${}_aR$, $a = 1,\cdots,n$, be as above. Then each ${}_aR$ preserves ${}^{\mathcal{F}}\mathcal{O}(M)$.*

Proof. Take $i > p, j \leqslant p$, and pick $r_0 \in {}^{\mathcal{F}}\mathcal{O}(M)$, so that by (5), $e^i_j(t=0) = 0$. We need to show that $e^i_j(t) = 0$ for all t. The right-hand side of the second equation in (12) is zero at $t = 0$ since $e^l_j(t=0) = 0$ unless $l \leqslant p$, and by (10), ${}^{\oplus}\Gamma^{i>p}_{k,l\leqslant p} \equiv 0$. According to the theory of first-order differential equations, if a flow starts at a point in a closed submanifold $N_1 \subset N$ and the vector field is tangent to N_1 at every point in N_1, then the flow stays in N_1; taking N to be $\mathcal{GL}(M)$ and N_1 to be ${}^{\mathcal{F}}\mathcal{GL}(M)$, the bundle of all frames with first p vectors along $\mathcal{F}$, we see that $e^{i>p}_{j\leqslant p}(t) = 0$ for all t. Thus each flow ${}_aR(t,\cdot)$ takes ${}^{\mathcal{F}}\mathcal{GL}(M)$ to itself. Moreover, the vector fields Y_a are horizontal for the connection $\nabla^{\oplus}$, and (12) says precisely that each tangent vector $e_j(t)$ is parallel along the curve $t \mapsto z(t)$. But parallel transport along $z(\cdot)$ preserves the metric g because $\nabla^{\oplus}$ does; hence the ${}_aR$ also preserve $\mathcal{O}(M)$. Therefore, they preserve ${}^{\mathcal{F}}\mathcal{O}(M) = \mathcal{O}(M) \cap {}^{\mathcal{F}}\mathcal{GL}(M)$. □

The following immediate corollary deals with constant linear combinations of the flows ${}_aR$. The flow ${}_aR$ constructed in Lemma 3.1 corresponds to the case $\vec{c} = E_a \in \mathbb{R}^n$.

Corollary 3.2. *Consider the flow $R(t,\cdot,\vec{c})$ given by the vector field $Y = \sum_1^n c_i Y_i$, where the c_i are constants. Then R preserves ${}^{\mathcal{F}}\mathcal{O}(M)$.*

The next lemma is our main technical result. Because Lemma 2.1 is not valid unless $X \in T_z\mathcal{F}^{\perp}$, we must limit ourselves here to transverse flows $R(t,\cdot,\vec{c})$, those for which the first p components $c_i, 1 \leqslant i \leqslant p$, of $\vec{c}$ are zero.

Lemma 3.3. *Let $R(t,\cdot,\vec{c})$ be a transverse flow. Then in the notation of (9), if $r_0 \sim r_1 \bmod O(p)$ we have*

$$R(t,r_0,\vec{c}) \sim R(t,r_1,\vec{c}) \bmod O(p).$$

In particular, $\pi(R(t,r_0,\vec{c})) \sim \pi(R(t,r_1,\vec{c}))$, so R is adapted to $\mathcal{F}$.

Proof. We give the proof in several steps, proceeding from local to global.

1. The flow $R(t,\cdot,\vec{c})$ is defined by $Y = \sum_{a>p}^{n} c_a Y_a$. Thus

$$Y = c_a e_a^m \partial_m - {}^{\oplus}\Gamma_{kl}^m\, c_a e_a^k e_j^l \partial/\partial e_j^m,$$

where repeated indices are summed; $p+1 \leqslant a \leqslant n, 1 \leqslant m \leqslant n$, and so on. Let us write $X(t) = \sum_{a>p}^{n} c_a e_a(t)$, with m-th component $X^m(t) = \sum_{a>p}^{n} c_a e_a^m(t)$.

According to (12), the equations for the flow in local coordinates read:

$$(*) \qquad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} e_j^m(t) = -{}^{\oplus}\Gamma_{kl}^m(z(t)) c_a e_a^k(t) e_j^l(t) \\ &\frac{\mathrm{d}}{\mathrm{dt}} z^m(t) = c_a e_a^m(t). \end{aligned}$$

We must show that $\pi \circ R(t,r_0)$ respects $\mathcal{F}$.

Since π_* kills the vertical directions and takes $\sum_{m=1}^{p} \sum_a c_a e_a^m(t) \partial/\partial z_m$ to $T\mathcal{F}$, we need only check for each $m > p$ that $\sum_a c_a e_a^m(t,r_0) \frac{\partial}{\partial z^m}$ is foliate. That is, for each $m, a > p$ there must be no dependence of $e_a^m(t,r_0)$ on r_0 when r_0 varies locally along a leaf $\widetilde{\mathcal{L}} = \{r = [z,\vec{e}] \mid z \in \mathcal{L}, r = \mathbf{gs}(r_{\mathrm{ref}})\}$ (by varying locally, we mean that $z_0 = \pi(r_0)$ remains within the chart U).

Thus we need to examine the above system of ordinary differential equations for $m > p$. Here our choice of the connection $\nabla^{\oplus}$ is essential, as it allows us to effectively decouple the coordinates in C from those in A and B. First of all, by (10) it follows that the terms on the right-hand side are zero unless $l > p$, and since all frames are in ${}^{\mathcal{F}}\mathcal{O}(M)$, it follows that $j > p$ also, as otherwise $e_j^l(t) = 0$. In terms of the block decomposition in (5), the differential equations (*) for the

components in C yield the transverse system of equations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}e^{m>p}_{j>p}(t) &= -\sum_{k>p,l>p,a>p} {}^{\oplus}\Gamma^{m>p}_{kl}(z(t))c_a e^k_a(t)e^l_j(t)\\ &\quad -\sum_{k\leqslant p,l>p,a>p} {}^{\oplus}\Gamma^{m>p}_{kl}(z(t))c_a e^k_a(t)e^l_j(t)\\ &= -\sum_{l>p}\left(P^{\perp}\nabla_{X(t)}P^{\perp}\frac{\partial}{\partial z_l}\right)^m e^l_j(t)\\ &= -\sum_{l>p}\left(\nabla^T_{\overline{X(t)}}\overline{\frac{\partial}{\partial z_l}}\right)^{m-p}_{\overline{z(t)}} e^l_j(t),\\ \frac{\mathrm{d}}{\mathrm{dt}}z^m(t) &= \sum_{a>p}^{n} c_a e^m_a(t).\end{aligned} \tag{13}$$

In the first equality we have for emphasis separated out the terms with $k \leqslant p$; these correspond to the B components of $X(t) = \frac{\mathrm{d}}{\mathrm{dt}}z(t)$. In the second equality we have used $m > p$, so that $(P\nabla_{X(t)}P\frac{\partial}{\partial z_l})^m = 0$. The third equality follows from Lemma 2.1 and involves only the coordinates $\overline{z}, C$. Thus the connection $\nabla^{\oplus}$ has enabled us to split the C coordinates off from the A and B coordinates. By Lemma 2.2, the initial condition for $\overline{z}, C$ remains the same as r_0 varies in $\widetilde{\mathcal{L}}$. Hence the result follows since (13), taken for all $m > p$ and $j > p$, is a system of (nonlinear) first-order ordinary differential equations of the form $\frac{\mathrm{d}}{\mathrm{dt}}(\overline{z}(t), C(t)) = F(\overline{z}(t), C(t))$, where neither the initial condition nor F depends on the parameters x along the leaf $\widetilde{\mathcal{L}}$. The solution $\overline{z}(t), C(t)$ is therefore independent of $r_0 \in \widetilde{\mathcal{L}}$ for all times t provided the flow remains over U.

We conclude: Given frames $r_0, r_1 \in \widetilde{\mathcal{L}}$ with $z_0 = \pi(r_0), z_1 = \pi(r_1)$ in U, there exists $T > 0$ such that for all $t, 0 \leqslant t \leqslant T$, we have

$$C(R(t, r_0, \vec{c})) = C(R(t, r_1, \vec{c}))$$

and

$$\pi(R(t, r_0, \vec{c})) \sim \pi(R(t, r_1, \vec{c})).$$

By the definition of the lifted foliation $\widetilde{\mathcal{F}}$, these two facts imply that

$$R(t, r_0, \vec{c}) \sim R(t, r_1, \vec{c}) \bmod O(p) \text{ for all } t, 0 \leqslant t \leqslant T.$$

We note that in addition to the transverse component which is well under control, the flow R also has vertical and longitudinal components about which less can be said. Because of the vertical component, even if r_0 and r_1 lie on

the same leaf $\widetilde{\mathcal{L}}$, after a time t we have only $R(t, r_0, \vec{c}) \sim R(t, r_1, \vec{c}) \bmod O(p)$; however, the vertical component is of no consequence after we project by π. The longitudinal component, which for transverse flows is due to the bending of the leaves, on the other hand causes a drift along the leaves even after projection, and we must treat it together with the transverse motion in what follows.

2. Suppose next that $r_0 \sim r_1 \bmod O(p)$ and r_0 lies on a leaf $\widetilde{\mathcal{L}}$; then $r_1 \cdot \gamma =: \widehat{r}_1 \in \widetilde{\mathcal{L}}$ for some $\gamma \in O(p)$. Let τ be a path in $\widetilde{\mathcal{L}}$ joining r_0 and $\widehat{r}_1$. We continue to work locally and assume that the projection of τ under π is contained in U. By part 1),

$$R(t, r_0, \vec{c}) \sim R(t, \widehat{r}_1, \vec{c}) \bmod O(p).$$

On the other hand, the system (12) now reads, with Y_a replaced by $Y = \sum_{i=p+1}^{n} c_i Y_i$:

$$\begin{aligned} \frac{dz}{dt} &= \sum c_i\, e_i, \\ \nabla^{\oplus}_{\dot{z}(t)} \vec{e}(z) &= 0, \end{aligned}$$

where $R(0) = r_0 = [z_0, \vec{e}_0]$ and $i = p+1, \cdots, n$. Since for $h \in G = O(p) \times O(q)$ arbitrary we have $\sum_i (h^{-1}\vec{c})_i (\vec{e}h)_i = \sum_{i,j,k} h^{-1}_{ij} c_j h_{ki} e_k = \sum_k c_k e_k$, it is immediate from the form of this equation that

$$R(t, r \cdot h, \vec{c}) = R(t, r, h^{-1} \cdot \vec{c})\, h, \tag{14}$$

where $h^{-1} \cdot \vec{c}$ denotes ordinary multiplication of the vector $\vec{c}$ by the matrix h^{-1}. This argument holds equally well for unrestricted $\vec{c} \in \mathbb{R}^n$ and also establishes Eq. (19) below. Taking $h = \gamma$, it follows that

$$R(t, \widehat{r}_1, \vec{c}) = R(t, r_1 \cdot \gamma, \vec{c}) = R(t, r_1, \gamma^{-1} \cdot \vec{c}) \cdot \gamma.$$

Since $\gamma^{-1} \in O(p)$, we have $c_j = (\gamma^{-1} \cdot \vec{c})_j,\ j = p+1, \cdots, n$. Thus the transverse part (13) of the system of equations is not changed by the action of γ, so

$$R(t, r_1, \gamma^{-1} \cdot \vec{c}) \sim R(t, r_1, \vec{c}) \bmod O(p)$$

is clear. We conclude that there exists $T > 0$ such that $R(t, r_0, \vec{c}) \sim R(t, r_1, \vec{c}) \bmod O(p)$ for all $t, 0 \leqslant t \leqslant T$.

3. Next let $r_0 \sim r_1 \bmod O(p)$, with no restriction that $\pi(r_1)$ be in U. As before, we have $r_1 \cdot \gamma =: \widehat{r}_1 \in \widetilde{\mathcal{L}}$ for some $\gamma \in O(p)$. Let τ be a path in $\widetilde{\mathcal{L}}$ joining r_0 and $\widehat{r}_1$. We subdivide τ into segments, each of which projects under π into some

simple chart, and apply step 2) to each segment. We conclude that given r_0 and r_1 with $r_0 \sim r_1 \bmod O(p)$, there exists $T > 0$ such that

$$R(t, r_0, \vec{c}) \sim R(t, r_1, \vec{c}) \bmod O(p)$$

for all $t, 0 \leqslant t \leqslant T$.

4. Finally, let $r_0, r_1 \in {}^{\mathcal{F}}\mathcal{O}(M)$ with $r_0 \sim r_1 \bmod O(p)$ be arbitrary and define T_0 to be the supremum of all $t \geqslant 0$ such that

$$R(t, r_0, \vec{c}) \sim R(t, r_1, \vec{c}) \bmod O(p). \tag{15}$$

We claim that $T_0 = \infty$. If this is not so, then by the continuity of the flow R we may replace t by T_0 in (15). Applying part 3) to R with initial frames $r_0' = R(T_0, r_0, \vec{c})$ and $r_1' = R(T_0, r_1, \vec{c})$, and using the group property of the flow: $R(t+s, r) = R(t, R(s, r))$, we see that (15) holds for all t between 0 and some T_1 strictly greater than T_0, contrary to the definition of T_0. ☐

Thus the transverse deterministic flows $R(t, r, \vec{c})$ constructed above preserve ${}^{\mathcal{F}}\mathcal{O}(M)$ and are adapted to the foliation $\mathcal{F}$. We next pass to the transverse stochastic flow in the usual way by considering a dyadic decomposition $D_k,\ k = 1, 2, \cdots$, of the positive time axis into intervals $I_n = \{t \mid n/2^k \leqslant t < (n+1)/2^k\},\ n = 0, 1, \cdots$, and imagining that the coefficients c_i are randomly changed at times of the form $t_n = n/2^k$. By Lemma 3.3, the resulting flow $R(t, \cdot)$, with the coefficients c_i reshuffled in this way, again preserves ${}^{\mathcal{F}}\mathcal{O}(M)$ and is adapted to $\mathcal{F}$. It is possible to make sense of the limit as $k \to \infty$, and the result is called a stochastic flow.

More precisely, consider the stochastic differential equation

$$dR_t = Y_i(R_t) dw_t^i, \qquad R(0) = r_0, \tag{16}$$

where all differentials are understood in the Stratonovich sense, and the $w^i,\ i = p+1, \cdots, n$, are the components of a standard q-dimensional Brownian process W on $\mathbb{R}^q$. W lives on (Ω_q, P_0^W), the space of all continuous paths $\omega : [0, \infty] \to \mathbb{R}^q$ starting at 0, with the standard Wiener measure P_0^W. It is known that almost everywhere (with respect to P_0^W), each component w^i is Hölder continuous for any exponent $\alpha < 1/2$, but is differentiable almost nowhere.

There is a standard way to approximate the solution of (16) which involves replacing the Stratonovich differentials in Eq. (16) by a "polygonal approximation"

on dyadic intervals:

$$dR_t^{(k)} = \sum_{i=p+1}^{n} Y_i(R_t^{(k)})\dot{w}^{i,k}\, dt, \quad R^{(k)}(0) = r_0, \tag{17}$$

where

$$\dot{w}^{i,k}(t) = 2^k \left(w^i(t_k^+) - w^i(t_k) \right),$$

with $t_k \equiv [2^k t]/2^k$, $t_k^+ \equiv [1 + 2^k t]/2^k$. These are ordinary differential equations on the frame bundle with coefficients $c_i = \dot{w}^{i,k}$ constant on each dyadic interval, and their integral curves define a flow of diffeomorphisms.

It is a fact that the sequence of maps $R^{(k)}(t, r_0, \omega)$ converges in probability to the solution $R(t, r_0, \omega)$ of Eq. (16), uniformly on compact sets. Moreover, this convergence is actually in the C^m topology; hence there exists a subsequence $R^{(k)}(t, r_0, \omega)$ of these diffeomorphisms which converge, together with their derivatives with respect to r_0, to the limit map $R(t, r_0, \omega)$, for almost every ω with respect to P_0^W. For this and related results, we refer to [Bi, Chap. 1: Th. 2.1, Th. 4.1, and Th.1, p. 71].

It follows that the limit stochastic process R_t will inherit any properties of the approximating flows $R_t^{(k)}$ that persist under closure. In particular, using Lemmas 3.1 and 3.3 the transverse stochastic flow (16) constructed from the globally defined vector fields Y_i will be shown to preserve the adapted frame bundle ${}^{\mathcal{F}}\mathcal{O}(M)$ and respect the foliation $\mathcal{F}$.

The flow (16) does not drop to a flow on M, because of the dependence on the choice of frame r_0 above $z_0 \in M$. Nevertheless, the associated (transverse) transition semigroup T_t, defined on functions $f \in C(M)$ by

$$(T_t f)(z) = E[(f \circ \pi)(R(t, r, \cdot))] = \int_{\Omega_q} f\left(\pi(R(t, r, \omega))\right) P_0^W(d\omega), \tag{18}$$

is independent of the choice of frame $r \in {}^{\mathcal{F}}\mathcal{O}(M)$ over z. This is because the flow is equivariant:

$$R(t, r \cdot \gamma; \omega) = R(t, r; \gamma^{-1} \cdot \omega) \cdot \gamma \tag{19}$$

cf. [IW, Chap. V, Eq. (5.7)]. Indeed, the transformation $\omega \mapsto \gamma \cdot \omega$, $(\gamma \cdot \omega)^i = \gamma^i_j \omega^j$, leaves Wiener measure unchanged, so that the probability law of the projection $Z(t, z; \cdot) := \pi \circ R(t, r; \cdot)$ is independent of the choice of frame $r \in {}^{\mathcal{F}}\mathcal{O}(M)$ above $z \in M$. Only this law, not the projected "flow" itself, is relevant in (18).

Lemma 3.4. *For almost every ω, the transverse stochastic flow $R(t,\cdot,\omega)$ preserves ${}^{\mathcal{F}}\mathcal{O}(M)$ and is adapted to the foliation $\mathcal{F}$. In fact, there exists a P_0^W-negligible set N such that for all $t \geqslant 0$ and $\omega \notin N$*

$$R(t,r_0,\omega) \sim R(t,r_1,\omega) \bmod O(p) \text{ whenever } r_0 \sim r_1 \bmod O(p). \tag{20}$$

Proof. We will need the case $m = 0$ of the following result [Bi, Th. 2.1]:

There exists a subsequence n_k and a subset $N \subset \Omega$ with $P_0^W(N) = 0$ such that for all $\omega \notin N$,

$$R^{(n_k)}(t,\cdot,\omega) \text{ converges to } R(t,\cdot,\omega)$$

in the C^m topology, uniformly on compact subsets of $\mathbb{R}^+ \times {}^{\mathcal{F}}\mathcal{O}(M)$. The approximations $R^{(k)}$ appearing here are the ones defined by (17). In what follows we fix such a subsequence and for simplicity write k for n_k. That ${}^{\mathcal{F}}\mathcal{O}(M)$ is preserved for all $\omega \notin N$ is clear, since each approximation $R^{(k)}(t,\cdot,\omega)$ preserves ${}^{\mathcal{F}}\mathcal{O}(M)$ and ${}^{\mathcal{F}}\mathcal{O}(M)$ is closed in $\mathcal{GL}(M)$.

Clearly, adaptedness is implied by (20), so it suffices to prove the latter. This follows from our previous results, which imply that the approximations (17) satisfy (20). Indeed, Lemma 3.3 applies and it is enough to consider a composition $\Psi \circ \Phi$ of two diffeomorphisms, where

$$\Phi = R(t,\cdot) \text{ and } \Psi = R'(t',\cdot),$$

with $t = 1/2^k$ and t' satisfying $0 \leqslant t' \leqslant 1/2^k$. This composition corresponds to running (17) from time zero to time $1/2^k + t'$, with initial point $r_0 \in {}^{\mathcal{F}}\mathcal{O}(M)$; the flow R' is obtained by reshuffling at time $t = 1/2^k$ the coefficients c_i determining R, as described after the proof of Lemma 3.3. By Lemma 3.3 applied to $Y = \sum c_i Y_i$, where the c_i are the constants for the flow R, we see that $\Phi(r_0) \sim \Phi(r_1) \bmod O(p)$. Now apply Lemma 3.3 again, this time to the reshuffled flow R' with initial conditions $\Phi(r_0)$ and $\Phi(r_1)$, to conclude that $\Psi(\Phi(r_0)) \sim \Psi(\Phi(r_1)) \bmod O(p)$ and the approximating flows $R^{(k)}$ satisfy (20). In particular, $\pi(\Psi(\Phi(r_0))) \sim \pi(\Psi(\Phi(r_1)))$, so they are adapted to $\mathcal{F}$.

Finally, we need to show that the limit stochastic flow (16) on ${}^{\mathcal{F}}\mathcal{O}(M)$ satisfies (20). This is not automatic, because the leaves need not be closed. Let $r_0 \sim r_1 \bmod O(p)$ and repeat the proof of Lemma 3.3, joining r_0 to $\widehat{r}_1$ by a path τ in $\widetilde{\mathcal{L}}$. For fixed $t \geqslant 0$ and $\omega \notin N$ let us write Φ for the diffeomorphism $R(t,\cdot,\omega)$ of ${}^{\mathcal{F}}\mathcal{O}(M)$. Subdividing τ into small pieces and arguing on each piece, we may suppose that τ is contained in a plaque in a simple chart $\widetilde{U}$ and that the image of τ under $\pi \circ \Phi$ is contained in some simple chart U with distinguished coordinates $z = (x,y)$. Since the $R^{(k)}(t,r,\omega)$ converge to Φ uniformly in $r \in {}^{\mathcal{F}}\mathcal{O}(M)$ for all

$\omega \notin N$, for all sufficiently large k we have $\pi \circ R^{(k)}(t,r,\omega) \subset U$ for $r \in \tau$. As shown in the previous paragraph, each $\pi \circ R^{(k)}(t,\cdot,\omega)$ takes plaques in ${}^{\mathcal{F}}\mathcal{O}(M)$ to plaques in M, hence on taking limits we see that $\pi \circ \Phi(\tau)$ is contained in a plaque. Moreover, the C coordinates of $R(t,r_0,\omega)$ and $R(t,\widehat{r}_1,\omega)$ coincide, since by the first part of this proof this is true for the approximating flows $R^{(k)}(t,\cdot,\omega)$. From the definition of $\widetilde{\mathcal{F}}$ (as in the proof of Lemma 3.3), it follows that

$$R(t,r_0,\omega) \sim R(t,\widehat{r}_1,\omega) \bmod O(p) \text{ for almost every } \omega. \tag{21}$$

To finish, we observe that $r_1 = \widehat{r}_1 \cdot \gamma$ for some $\gamma \in O(p)$. Arguing as in the proof of Lemma 3.3, but using Eq. (19) in place of (14), we obtain from (21) that $R(t,r_0,\omega) \sim R(t,r_1,\omega) \bmod O(p)$, a.e. ω. □

In particular, $R(t,\cdot,\cdot)$ is weakly adapted to $\mathcal{F}$, and hence T_tf given by (18) is basic whenever f is.

The next lemma establishes an important property of the transition semigroup T_t when g is replaced by another bundle-like metric g'. We write ${}^{\mathcal{F}}\mathcal{O}(M)$ and ${}^{\mathcal{F}}\mathcal{O}(M)'$ for the adapted orthonormal frame bundles for g and g', respectively; the corresponding transverse transition semigroups are denoted by T_t and T'_t. Recall that as remarked after Eq. (18), for $f \in C(M)$, $T_tf(z) = E[f(\pi R(t,r_0,\cdot))]$ and $T'_tf(z) = E[f(\pi R'(t,r'_0,\cdot))]$ do not depend on the choice of the initial frames $r_0 \in {}^{\mathcal{F}}\mathcal{O}(M)$ and $r'_0 \in {}^{\mathcal{F}}\mathcal{O}(M)'$ over $z \in M$.

Lemma 3.5. *For all $z \in M$, we have*

$$T_tf(z) = T'_tf(z) \tag{22}$$

for all basic functions f.

Proof. By (18), (19), and the comment just before Lemma 3.4, we may replace the initial frame $r'_0 \in {}^{\mathcal{F}}\mathcal{O}(M)'$ by $r'_0 \cdot \gamma$, $\gamma \in G = O(p) \times O(q)$. By (3), we can choose $\gamma \in O(q)$ so that, in the notation of (5), the frame coordinates C'_0 for $r'_0 \cdot \gamma$ coincide with C_0 for r_0.

We begin by arguing locally within a coordinate chart U_1. Recalling (13), we get the transverse systems of differential equations for the two transverse deterministic flows R and R' in local coordinates:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} e^{m>p}_{j>p}(t) &= - \sum_{l>p, a>p, k} {}^{\oplus}\Gamma^{m>p}_{kl}(z(t)) c_a e^k_a(t) e^l_j(t) \\
&= - \sum_{l>p} \left(P^{\perp} \nabla_{X(t)} P^{\perp} \frac{\partial}{\partial z_l} \right)^m e^l_j(t), \qquad (23)\\
\frac{\mathrm{d}}{\mathrm{dt}} z^m(t) &= X^m(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} {e'}^{m>p}_{j>p}(t) &= - \sum_{l>p, a>p, k} {}^{\oplus}{\Gamma'}^{m>p}_{kl}(z'(t)) c_a {e'}^k_a(t) {e'}^l_j(t) \\
&= - \sum_{l>p} \left({P'}^{\perp} \nabla_{X'(t)} {P'}^{\perp} \frac{\partial}{\partial z_l} \right)^m {e'}^l_j(t), \qquad (24)\\
\frac{\mathrm{d}}{\mathrm{dt}} {z'}^m(t) &= {X'}^m(t)
\end{aligned}
$$

In writing (24) we use the direct-sum connection $\nabla^{\oplus'}$ for the metric g' on M and the associated canonical vector fields Y_i'; ${P'}^{\perp}$ is the orthogonal projection on $(T\mathcal{F})^{\perp}$ for g'. Recall that $X(t) = \sum_{a>p}^n c_a e_a(t)$, and we define similarly $X'(t) = \sum_{a>p}^n c_a e_a'(t)$.

By Lemma 2.1, we have (as in the first part of the proof of Lemma 3.3)

$$
\begin{aligned}
\overline{P^{\perp} \nabla_{X(t)} P^{\perp} \frac{\partial}{\partial z_{l>p}}} &= \nabla^T_{\overline{X(t)}} \overline{\frac{\partial}{\partial z_l}} \quad (\text{at } \overline{z}(t)) \\
\overline{{P'}^{\perp} \nabla'_{X'(t)} {P'}^{\perp} \frac{\partial}{\partial z_{l>p}}} &= \nabla^T_{\overline{X'(t)}} \overline{\frac{\partial}{\partial z_l}} \quad (\text{at } \overline{z'}(t)),
\end{aligned}
$$

where ∇^T denotes the Levi-Cività connection for the transverse metric g_T on the local model space $\overline{M/\mathcal{F}}$. Thus the form of the two equations (23), (24) for the coordinates $(\overline{z}, C)$ and $(\overline{z'}, C')$ is identical; since the initial conditions coincide, we see that $(\overline{z(t)}, C(t)) = (\overline{z'(t)}, C'(t))$.

Next, we must globalize this result. The difficulty is that although the transverse parts of g and g' are the "same" by (3), there is no correlation in the variation of the longitudinal parts of g and g' as we move along a leaf. This results in a longitudinal drift of the two flows relative to one another which must be treated here.

Fix some time $t > 0$ such that for all $0 \leqslant \tau \leqslant t$, both $\pi(R(\tau))$ and $\pi(R'(\tau))$ lie within the chart U_1, while $\pi(R'(t))$ also lies in an overlapping chart U_2. The

initial frames for R, R' are $r_0 \in {}^{\mathcal{F}}\mathcal{O}(M)$ and $r_0' \in {}^{\mathcal{F}}\mathcal{O}(M)'$. Before starting up the flows, we were free to replace r_0' by $r_0' \cdot \gamma,\ \gamma \in O(q)$, so that its initial C coordinates C' agreed with those of r_0. As the flows evolve in time, however, it is essential that we not do this again as this would change the transverse equations (24) for $R'(t)$, which is not allowed.

By the part of Lemma 3.5 already proved, we have

$$C'(t) = C(t) \tag{25}$$

using the coordinates in the chart U_1, and the projections $z_t = \pi(R_t)$ and $z_t' = \pi(R_t')$ lie on the same leaf $\mathcal{L}_t$ of $\mathcal{F}$. (Here we write R_t for $R(t, r_0)$ and similarly for R_t'.) Let σ be a path in $\mathcal{L}_t \cap U_1$ from z_t to z_t' and let $\widetilde{\sigma}$ be the lift of σ starting at R_t and contained in $\widetilde{\mathcal{L}}_t$. The endpoint A_t of $\widetilde{\sigma}$ satisfies $\pi(A_t) = z_t' = \pi(R_t')$. Let ${}^{\mathrm{tr}}R : s \mapsto R(s, A_t)$ denote the "translated" flow with initial value $A_t, 0 \leqslant s$. By Lemma 2.2 applied to the metric g, bundle ${}^{\mathcal{F}}\mathcal{O}(M)$, and lifted foliation $\widetilde{\mathcal{F}}$,

$$C(A_t) = C(t)$$

because σ lies within the chart U_1. Thus, by Eq. (25) we have

$$C(A_t) = C'(t) \tag{26}$$

in terms of the coordinates for the chart U_1, and therefore also in terms of the coordinates in the overlapping chart U_2 (recall the discussion after Lemma 2.2).

The essential point is that by Eq. (26), the new initial points R_t' and A_t are already "in register" in terms of the coordinates of chart U_2, so no further application of $\gamma \in O(q)$ is necessary. Letting the flows develop from $A_t = {}^{\mathrm{tr}}R(s = 0)$ and $R'(0, R_t')$ for a time $s > 0$ small enough so that we remain in U_2, we obtain (using the semigroup property of the flows and the notation of (9)):

$$\pi(R_{t+s}) \sim \pi({}^{\mathrm{tr}}R_s) \sim \pi(R_{t+s}').$$

The first relation holds by Lemma 3.3 applied to R, and the second follows by an another application of the first part of the proof of Lemma 3.5, this time within the chart U_2.

Thus we can use Lemma 3.3 to translate the flow R_t along $\widetilde{\mathcal{F}}$, compare the translated flow with R_t' in some other chart, and deduce that $\pi(R_t) \sim \pi(R_t')$ for all times $t \geqslant 0$.

The next step is to treat the approximating flows $R^{(k)}(t, \cdot)$ in (17), which is done by considering composites of flows corresponding to vector fields $Y = \sum c_i Y_i$ with initial conditions $r_0 \in \widetilde{\mathcal{L}} \cdot O(p)$. The argument is the same as in the proof of Lemma 3.4.

Thus the approximating flows satisfy $\pi(R_t^{(k)}) \sim \pi(R_t'^{(k)})$ for all $t \geqslant 0$, and the analogous result for the stochastic flows holds for almost every ω on passing to the limit. The equality (22) now follows from (18). $\square$

4. EXTENSION TO FORMS

Let u be a tensor of type (a, b). In terms of the local coordinates $z_1, \cdots, z_n$, $u(z)$ is given in terms of its components $u(z)_L^K$ by

$$u(z) = u(z)_L^K \partial_K \otimes dz^L,$$

where $K = (k_1, \cdots, k_a)$ and $L = (l_1, \cdots, l_b)$ are multi-indices of degree a and b; $\partial_K \equiv \frac{\partial}{\partial z_{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial z_{k_a}}$ and $dz^L \equiv dz^{l_1} \otimes \cdots \otimes dz^{l_b}$.

In terms of frames $r = [z; \vec{e}]$ we can write

$$u(z) = F_{uJ}^I(r)\, e_I \otimes e_*^J = F_{uJ}^I(r)\, e_I^K\, f_L^J\, \partial_K \otimes dz^L, \tag{27}$$

where I, J are multi-indices, and $e_I \equiv e_{i_1} \otimes \cdots \otimes e_{i_a}$, and so on. The coordinates e_k^i, f_i^k of the k-th frame vector e_k and the k-th vector e_*^k of the dual frame are defined by

$$e_k = e_k^i \frac{\partial}{\partial z_i}, \qquad e_*^k = f_i^k dz^i; \tag{28}$$

the matrix (f_i^j) is the inverse of (e_i^j). If $r = [z; \vec{e}]$ is expressed in block form as in Eq. (5), then

$$(e_j^i) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \qquad \text{and} \qquad (f_j^i) = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

The functions F_{uJ}^I are well-defined on the entire frame bundle; however, the components e_I^K, f_L^J in (27) are defined only with reference to the local chart $\{z_j\}$. Observe that the definition (28) for e_*^k involves the transpose of (f_j^i); thus we regard e_k as the k^{th} column vector of (e_j^i) and e_*^k as the k^{th} row vector of (f_j^i). The e_k with $1 \leqslant k \leqslant p$ span $T\mathcal{F} = \text{span}\{\partial/\partial z_i\}, 1 \leqslant i \leqslant p$, while the e_*^k with $p+1 \leqslant k \leqslant n$ span the transverse space $Q^* = \text{span}\{dz^a\}, p+1 \leqslant a \leqslant n$.

The collection of functions $\{F_{uJ}^I\}$ on the frame bundle is called the *scalarization* of u and is equivariant (see, e.g., [IW, p. 280] or [BGV, p. 24]). That is,

$$F_{u\cdot}^{\cdot}(r \cdot \gamma) = F_{u\cdot}^{\cdot}(r) \cdot (\gamma^{\otimes})^{-1}, \tag{29}$$

where $r \cdot \gamma$ is given by (8).

Conversely, if (29) holds for some collection $\{F^I_J\}$ of functions, then there exists a unique tensor u of which $\{F^I_J\}$ is the scalarization. We have

$$u(z)^K_L = F^I_{uJ}(r) e^K_I f^J_L, \qquad F^I_{uJ}(r) = u(z)^K_L e^L_J f^I_K. \tag{30}$$

We now specialize to the case when $u = \theta(z) = \theta(z)_J \, dz^J$ is an m-form and consider only frames $r \in {}^{\mathcal{F}}\mathcal{O}(M)$.

Lemma 4.1. *θ is basic if and only if:*

i) each $F_{\theta J}$ is constant along $\widetilde{\mathcal{L}} \cdot O(p)$ ($\widetilde{\mathcal{L}}$ a leaf of $\widetilde{\mathcal{F}}$) and

ii) $F_{\theta J}(r) = 0$ whenever any index $j_\nu \leqslant p$.

In other words, θ is basic if and only if the $F_{\theta J}$ depend only on the C coordinates for $J > p$ and vanish otherwise.

Proof. The straightforward proof [Ma] is based on Lemma 2.2. □

Given a form θ with scalarization $\{F_{\theta J}\}$, we set

$$U_J(t, r_0) = E[F_{\theta J}(R(t, r_0, \omega))] \equiv \int_{\Omega_q} F_{\theta J}(R(t, r_0, \omega)) P^W_0(d\omega). \tag{31}$$

By (19), the transverse flow R is $G = O(p) \times O(q)$-equivariant. Since $\{F_{\theta J}(\cdot)\}$ is equivariant (29), the same is true of $\{U_J(t, \cdot)\}$ for each $t \geqslant 0$, because $\omega \mapsto \gamma \cdot \omega$ leaves the measure P^W_0 unchanged. By the observation made after (29), it follows that there exists a unique m-form $\theta(t, z_0)$ of which $\{U_J(t, r_0)\}$ is the scalarization. The action of the transverse semigroup T_t on forms is defined by

$$(T_t\theta)(z) = \theta(t, z). \tag{32}$$

We have

Lemma 4.2. *$T_t\theta$ is basic whenever θ is.*

Proof. This follows from Lemmas 3.4 and 4.1. □

We note here that the extension (32) of T_t to differential forms is easily seen to preserve the filtration (4).

5. THE HEAT EQUATION

We now consider, in addition to the transverse semigroup T_t constructed above, the full semigroup S_t constructed as in (18), but using the full stochastic flow $R(t, r, \omega)$ constructed as described after Lemma 3.3 from the unrestricted deterministic flows $R(t, r, \vec{c})$, for which $\vec{c} \in \mathbb{R}^n$ is arbitrary; thus in (18), Ω_q is replaced by Ω_n. The infinitesimal generator of S is elliptic, as required for strict positivity of the heat kernel and ergodicity, which we need in Section 6. However, because the full flow does not respect the foliation, it is not clear that S_t preserves the basic functions, though this crucial property holds for T_t (Lemma 3.4). Nevertheless, it is a remarkable fact that after the averaging over n-dimensional Wiener measure is performed to get S we have $S_t f = T_t f$ for all basic functions f. In the present section we prove this result and examine some properties of the infinitesimal generators.

We begin by recalling the fundamental result [IW, Chap. V, Th. 3.1] that the transition semigroups T_t and S_t defined by (18) give solutions to the heat equation. Namely, set $\widetilde{\nu}_f(t, r) = S_t f(t, r) \equiv E[f(R(t, r, \cdot))]$ for any $f \in C^\infty({}^{\mathcal{F}}\mathcal{O}(M))$; then $\widetilde{\nu}_f$ satisfies the partial differential equation

$$\frac{\partial \widetilde{\nu}_f}{\partial t} = \frac{1}{2} \sum_1^n Y_k^2 \, \widetilde{\nu}_f, \qquad \widetilde{\nu}_f(0, r) = f(r). \tag{33}$$

Let us write

$$\widehat{A} \equiv \frac{1}{2} \sum_1^n Y_k^2. \tag{34}$$

In the corresponding equation for the transverse semigroup T_t, $\widehat{A}$ is replaced by $\widehat{A^\perp}$, the summation over k now going from $p+1$ to n.

The proof of the next lemma is an application of [IW, Chap. V, Eq. (4.33)]; indeed, Ikeda and Watanabe show that any drift vector field $\vec{b}$ on M can be obtained by using a suitable affine connection ∇ on M that preserves the metric but has nonzero torsion in general [IW, Prop. V.4.3]. The direct sum connection $\nabla^\oplus$ used here preserves the metric, and we will now see that its torsion is such that the drift field $\vec{b}$ is just $\frac{1}{2}\kappa$, where κ is the mean curvature field.

Lemma 5.1. *For $f \in C^\infty(M)$, consider the lift $f \circ \pi$ to ${}^{\mathcal{F}}\mathcal{O}(M)$, and let $\widehat{A}$ be as in (34). Then*

$$\widehat{A}(f \circ \pi) = (Af) \circ \pi, \tag{35}$$

where

$$A = \frac{1}{2}\Delta_M + \frac{1}{2}\kappa. \tag{36}$$

Here $\Delta_M = -\delta d = +g^{ij}\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} - g^{ij}\Gamma^k_{ij}\frac{\partial}{\partial z_k}$ *is the Laplacian for the given bundle-like metric* g.

Proof. The drift field $\vec{b}$ is given in local coordinates by

$$b^i = \frac{1}{2}g^{km}\left(\Gamma^i_{km} - {}^{\oplus}\Gamma^i_{km}\right), \tag{37}$$

where Γ^i_{km} and ${}^{\oplus}\Gamma^i_{km}$ are the Christoffel components for the Riemannian and direct-sum connections, respectively. Moreover, (35) holds with $A = \frac{1}{2}\Delta_M + \vec{b}$, see [IW, Chap. V, Eq. (4.33)].

To show (36), pick $z \in M$ and a simple neighborhood $U \ni z$ in M with coordinates z_a, such that the $z_a = x_a$ with $1 \leqslant a \leqslant p$ are along $\mathcal{F}$ while the $z_b = y_{b-p}$, $p+1 \leqslant b \leqslant n$, are transverse. By definition, the mean curvature is the vector field given by

$$\kappa = \sum_{a=1}^{p}\sum_{b=p+1}^{n} g(\nabla_{e_a} e_a,\ e_b)\, e_b, \tag{38}$$

for any local orthonormal frame $\{e_i\}$ with e_a in $T\mathcal{F}$ and e_b in $(T\mathcal{F})^{\perp}$. We will take the e_i, $1 \leqslant i \leqslant n$, to be obtained by applying the Gram–Schmidt procedure to

$$\partial/\partial z_1, \cdots, \partial/\partial z_p, \partial/\partial z_{p+1}, \cdots, \partial/\partial z_n,$$

in the given order. We have seen that because the metric g is bundle-like, the e_i are foliate (recall the discussion preceding Lemma 2.1). Since the vector field $\vec{b}$ is tensorial, in (37) we can work with the local field of orthonormal frames $\{e_i\}$ just constructed and obtain

$$\begin{aligned} 2b^i &= \sum_k (\nabla_{e_k} e_k - {}^{\oplus}\nabla_{e_k} e_k,\ e_i) \\ &= \sum_{k \leqslant p} (e_i,\ P^{\perp}\nabla_{e_k} e_k) + \sum_{k>p} (e_i,\ P\,\nabla_{e_k} e_k). \end{aligned} \tag{39}$$

We consider the two cases $i > p$ and $i \leqslant p$ separately.

For $i > p$ we have $2b^i = \sum_{k \leqslant p} g(e_i,\ \nabla_{e_k} e_k) = \kappa^i$ by (38).

For $i \leqslant p$, (39) reduces to

$$2b^i = \sum_{k>p} g(e_i,\ \nabla_{e_k} e_k).$$

By the Koszul formula,

$$2g(\nabla_{e_k} e_k,\ e_i) = 2g(e_k, [e_i, e_k]),$$

which is zero because $e_{k>p}$ is foliate, i.e., $[e_{i\leqslant p}, e_k] \in T\mathcal{F}$. We conclude that $\vec{b} = \frac{1}{2}\kappa$. ☐

For $f \in C^\infty(M)$ and $z \in M$, let us write $\nu_f(t,z) \equiv \widetilde{\nu}_{f\circ\pi}(t,r) = E[f \circ \pi(R(t,r,\cdot))]$, where $\pi(r) = z$ and we are using the full flow R; by the discussion after (18) this is well-defined, i.e., independent of the choice of frame r over z. Since $\widetilde{\nu}_{f\circ\pi}(t,r) = \nu_f(t,\pi(r))$, it follows from equation (33), with f replaced by $f\circ\pi$, and the relation (35): $\widehat{A}(\nu_f \circ \pi) = (A\nu_f)\circ\pi$, that $\nu_f(t,z) = (S_t f)(z)$ satisfies the heat equation on M:

$$\frac{\partial \nu_f}{\partial t}(t,z) = A\nu_f(t,z), \qquad \nu_f(t=0,z) = f(z). \tag{40}$$

Lemma 5.2. *For every basic function f, we have $S_t f = T_t f$ for all $t \geqslant 0$. In particular, $S_t f$ is basic.*

Proof. We have $\frac{\mathrm{d}}{\mathrm{d}t} S_t f = AS_t f$ in general. Moreover, for *basic* f,

$$\begin{aligned}\frac{1}{2}\left((\Delta_M + \kappa) f\right)\circ\pi &= (Af)\circ\pi = \widehat{A}(f\circ\pi)\\ &= \frac{1}{2}\sum_{k=1}^{n} Y_k^2(f\circ\pi) = \frac{1}{2}\sum_{k=p+1}^{n} Y_k^2(f\circ\pi) = \widehat{A^\perp}(f\circ\pi),\end{aligned}$$

hence $\frac{\mathrm{d}}{\mathrm{d}t} T_t f = AT_t f$, where we have used the fact that $T_t f$ is basic for all t (Lemma 3.4). By uniqueness of solutions of the heat equation it follows that $S_t f = T_t f$. ☐

Corollary 5.3. *The differential operator $A = \frac{1}{2}\Delta_M + \frac{1}{2}\kappa$ leaves $C_b^\infty(M)$ invariant.*

Proof. Recall that $\nu_f(t,z) = (S_t f)(z)$ and we have seen that S_t preserves $C_{\mathrm{b}}(M)$. Thus for $f \in C_{\mathrm{b}}^\infty(M)$, each $\nu_f(t,\cdot)$ is basic and the result follows by setting $t = 0$ in (40). ☐

By considering the scalarizations (§4), we can derive a result for T_t acting on forms.

Theorem 5.4. *The infinitesimal generator of the transverse semigroup T_t acting on forms (32) is*

$$A = \frac{1}{2}\Delta^{\oplus},$$

where $\Delta^{\oplus}\theta = +\nabla^{\oplus}_{e_i}(\nabla^{\oplus}_{e_i}\theta) - \nabla^{\oplus}_{\nabla^{\oplus}_{e_i} e_i}\theta$, for any local orthonormal frame $\{e_i\}$ in ${}^{\mathcal{F}}\mathcal{O}(M)$ (summation on i from $p+1$ to n is understood). In particular, A preserves the basic complex.

Proof. The proof is analogous to that of the Cor. 5.3. Equation (33) now holds componentwise for each function in the scalarization $\{F_{\theta J}\}$ of θ. We need the fact that because Y_k is horizontal,

$$Y_k F_{\theta\, J}(r) = (F_{\nabla^{\oplus}\theta})_{J,k}(r). \tag{41}$$

This follows from a straightforward calculation, cf. Proposition 4.1 in [IW, Chap. V]. It also follows more conceptually from the commutative diagram

$$\begin{array}{ccc} C^{\infty}({}^{\mathcal{F}}\mathcal{O}(M), V^{\Lambda})^{G} & \xrightarrow{d+\rho^{\Lambda}_{*}(\omega.)} & \mathcal{A}^1({}^{\mathcal{F}}\mathcal{O}(M), V^{\Lambda})_{basic} \\ \alpha_0 \big\downarrow \wr & & \alpha_1 \big\downarrow \wr \\ \mathcal{A}^0(M, \mathcal{A}^j) & \xrightarrow{\nabla^{\oplus}} & \mathcal{A}^1(M, \mathcal{A}^j) \end{array} \tag{42}$$

for the case of j-forms (see, e.g., [BGV, p. 24]). In (42) $\mathfrak{g}$ is the Lie algebra of the structure group $G = O(p) \times O(q)$ of the principal bundle ${}^{\mathcal{F}}\mathcal{O}(M)$; $\mathfrak{g}$ acts by the differential ρ^{Λ}_{*} of the representation ρ^{Λ} of G on the vector space V^{Λ} built up by taking alternating tensor products of ρ_0, the dual of the standard representation of G on $V = \mathbb{R}^p \oplus \mathbb{R}^q$ (recall the discussion around (8)); $C^{\infty}({}^{\mathcal{F}}\mathcal{O}(M), V^{\Lambda})^{G}$ is the space of smooth G-equivariant maps; ω is the $\mathfrak{g}$-valued one-form (connection) corresponding to the covariant derivative $\nabla^{\oplus}$. The scalarization $\{F_{\theta\, J}\}$ in (27) gives the equivariant map in the upper left-hand corner of the diagram, cf. (29).

For the second-order derivatives appearing in (33) (with the lower limit $k = 1$ replaced by $k = p + 1$), Eq. (41) gives

$$Y_k Y_k F_{\theta\, J}(r) = (F_{\nabla^{\oplus}\nabla^{\oplus}\theta})_{J,k,k}(r). \tag{43}$$

From (32), (31), (43), and (33), with $\widetilde{\nu}_f$ replaced by $\{F_{\theta_t\, J}\}$, it follows that

$$\frac{\partial\theta_t}{\partial t} = \frac{1}{2}\Delta^{\oplus}\theta_t,$$

where $\theta_t \equiv T_t\theta$.

Arguing as in the proof of the above Corollary, but using this time Lemma 4.2, we see that A preserves the basic complex. $\square$

We close this section with a quick proof of the analog of Lemma 3.5 for forms.

Lemma 5.5. *Let $\theta \in \mathcal{A}_b(M)$ be a basic m-form and let g, g' be two bundle-like metrics satisfying (3). Then*

$$T_t\theta = T'_t\theta \text{ for all } t \geqslant 0.$$

Proof. We have from (32), (31), and the first equality in (27) that $T_t\theta(z) = \int_{\Omega_q} F_{\theta J}(R(t,r,\omega))P_0^W(d\omega)e_*^J(r)$ and $T'_t\theta(z) = \int_{\Omega_q} F_{\theta J}(R'(t,r',\omega))P_0^W(d\omega){e'}_*^J(r')$. By Lemma 4.1(ii), only multi-indices J with every component $> p$ appear in these equations. We again choose $r' \in {}^{\mathcal{F}}\mathcal{O}(M)'$ over $z \in M$ so that $C'(r') = C(r)$; thus $e_*^J(r) = {e'}_*^J(r')$. Lemma 4.1(i) now permits us to repeat the proof of Lemma 3.5 with $f \circ \pi$ replaced by $F_{\theta J}$. $\square$

Differentiating $T_t\theta = T'_t\theta$ at $t = 0$, we obtain $A\theta = A'\theta$ for all basic forms θ, where A, A' are given by Theorem 5.4 for the metrics g, g'. This result expresses a general invariance principle which would be cumbersome to prove directly.

Finally, let us remark that the dependence on the homotopy class of γ (i.e., covering-space phenomena associated with $\pi : \widetilde{\mathcal{L}} \to \mathcal{L}$) mentioned after Lemma 2.2 plays no role in this work. For functions, this is because the projection π appears in the definition (18) of T_t and S_t; for basic forms θ, it is because of Lemma 4.1(i).

6. THE FUNCTION ϕ

Because P_0^W is a probability measure, the transition semigroup S_t (18) acts by contractions on $C(M)$, the Banach space of continuous functions on M with the sup norm. The infinitesimal generator $A = \frac{1}{2}(\Delta_M + \kappa)$ acts on the smooth functions $C^\infty(M) \subset C(M)$ and is closable. The dual semigroup S_t^* acts on $C(M)^* = \mathrm{Meas}(M)$, the Banach space of real-valued (signed) measures on M, and its infinitesimal generator A^* is a closed, densely defined operator on $C(M)^*$. For $h \in C(M)$ smooth, A^*h is given by the formal adjoint of A:

$$A^*h = \frac{1}{2}\left(\Delta_M h - \mathrm{div}(h\kappa)\right) = -\delta(dh - h\kappa)/2. \tag{44}$$

Here we regard h as the measure $h\,\mathrm{dvol}_M$ on M, where dvol_M is the Riemannian volume element on M.

Since we can work separately with each connected component, there is no loss of generality in assuming M to be connected as well as compact. It is then well known that the transition semigroup S_t has a unique invariant probability measure (see, e.g., [IW, Prop. V.4.5], [Kun, Th. 1.3.6], [N]), and by elliptic regularity this measure is of the form $\phi\mathrm{dvol}_g$, with $\phi \geqslant 0$ smooth. We will need the fact that $\phi > 0$ everywhere.

Proposition 6.1. *Let M be compact and connected. Then there exists a unique probability measure $\mu(dz)$ invariant under S_t. It is given by $\phi\, dvol_M$, where $\phi \in C^\infty(M)$, $\phi > 0$ everywhere, and $A^*\phi = 0$, i.e.,*

$$0 = \delta(d\phi - \phi\kappa).$$

Proof. We sketch an argument [Ma]. Since A is elliptic with vanishing zero-order part, its kernel reduces to the constants. By the index theorem, $\mathrm{index}(A) = \mathrm{index}(\Delta) = 0$, hence $\dim \ker(A^*) = 1$. Choose $\phi \not\equiv 0$ with $A^*\phi = 0$; by elliptic theory, ϕ is smooth. The associated measure $\mu = \phi\,\mathrm{dvol}_g$ on M is invariant under the adjoint semigroup S_t^*, which like S_t is a positivity-preserving contraction. It then follows by a standard argument that we can take μ to be a positive measure, i.e., $\phi \geqslant 0$. If ϕ were to vanish at some point $z_0 \in M$, then writing out the equation $A^*\phi = 0$ in local coordinates and using the ellipticity of A^*, we see that all derivatives of ϕ through order two vanish at z_0. Repeatedly differentiating the equation $A^*\phi = 0$, setting $z = z_0$, and proceeding by induction, we find that all derivatives of ϕ vanish at z_0. Therefore, by Aronszajn's theorem $\phi \equiv 0$, a contradiction. Alternatively, the results of [Bo] could also be used to show that $\phi > 0$. $\square$

Definition 5. Let $\psi > 0$ be smooth, $p = \dim \mathcal{F}$. If g' is obtained from g by leaving $Q \equiv T\mathcal{F}^\perp$ unchanged while rescaling g along $T\mathcal{F}$ by $\psi^{2/p}$, so that $g' = \psi^{2/p} g_{\mathcal{F}} \oplus g|_Q$, we say that g' is an $\mathcal{F}$-dilation of g.

If g is bundle-like (satisfies (3)), then clearly so is g'.

Our immediate concern is with $\mathcal{F}$-dilations, for which we will need to consider the long-time behavior $t \to \infty$. Because the generator $A = \frac{1}{2}(\Delta_M + \kappa)$ of the transition semigroup S_t is not symmetric, we cannot argue as in the usual case of a self-adjoint negative generator A, where $\lim_{t\to\infty} e^{tA}\psi$ is the projection of the function or form ψ onto its harmonic part. But there is a substitute in the form

of the ergodic theorem ([Kun, Th. 1.3.10]). This holds for any Feller semigroup $\{S_t\}$ for which the transition probability $P_t(z, dw)$ is given by

$$P_t(z, dw) = p_t(z, w)\mathrm{vol}(dw) \tag{45}$$

for some strictly positive kernel $p_t(z, w)$ that is continuous in $(t, z, w) \in (0, \infty) \times M^2$. (We recall that the transition probability $P_t(z, dw)$ is the measure defined by the positive linear functional $f \mapsto S_t f(z)$, so that $S_t f(z) = \int_M f(w) P_t(z, dw)$.)

The Feller condition is easily established (see, e.g., [Ma]). A proof that the kernel $p(t, z, w) = p_t(z, w)$ exists and is continuous can be found in [BGV, Th. 2.23]. Since $S_t f(z) \geqslant 0$ for $f \geqslant 0$, we see that (45) holds with $p_t \geqslant 0$. To show that $p_t > 0$, one can apply the strong maximum principle; see, e.g., Theorem 3.1 in [Bo]. In fact, Bony's results hold quite generally for hypoelliptic operators and are thus more than we need here. In particular, strict positivity of the heat kernel for T_t itself would follow if the latter were hypoelliptic, but this is hardly ever the case for Riemannian foliations. So for technical reasons we work with S_t.

Thus the ergodic theorem applies to our situation and we conclude that for any $f \in C(M)$ and $z \in M$,

$$\lim_{t\to\infty} S_t f(z) = \int_M f\phi \,\mathrm{dvol}_g,$$

$\phi \,\mathrm{dvol}_g$ being the unique invariant probability measure on M given by Proposition 6.1.

We now dilate the bundle-like metric g by ϕ:

$$g' = \phi^{2/p} g_{\mathcal{F}} \overset{\perp}{\oplus} g_Q. \tag{46}$$

Then $\mathrm{dvol}_{g'} = \phi \,\mathrm{dvol}_g$.

The new transition semigroup is S'_t, and its infinitesimal generator A' is given by $A' = \frac{1}{2}(\Delta_{g'} + \kappa')$, where $\kappa' = \kappa - d_{1,0} \log \phi$ as follows from Rummler's formula (see for instance [Dom, Eq. (4.22)]). By Lemmas 3.5 and 5.2, for all basic functions f

$$S'_t f(z) = S_t f(z) \;\forall\, z \in M. \tag{47}$$

We note in passing that in the special case of dilations considered here it is not difficult to show directly that $A'f = Af$ for f basic, hence $S'_t f = S_t f$ follows by the same uniqueness argument as in the proof of Lemma 5.2, thus avoiding Lemma 3.5. However, Lemma 3.5 holds for arbitrary changes of metric subject to (3) and is useful in more general situations, as in Lemma 5.5.

Let us write $\phi' \,\mathrm{dvol}_{g'}$ for the unique probability measure on M invariant under S'_t; ϕ' is given by Prop. 6.1. For $f \in C_{\mathrm{b}}(M)$ basic and $z \in M$ arbitrary, an application of the ergodic theorem gives

$$\begin{aligned}\lim_{t\to\infty} S'_t f(z) &= \int_M f\phi' \,\mathrm{dvol}_{g'} \\ &= \int_M f\phi'_{\mathrm{b}'} \,\mathrm{dvol}_{g'} \\ &= \int_M f\phi'_{\mathrm{b}'}\phi \,\mathrm{dvol}_{g} \\ &= \int_M f\phi'_{\mathrm{b}'}\phi_{\mathrm{b}} \,\mathrm{dvol}_{g},\end{aligned}$$

and by (47) this is equal to

$$\begin{aligned}\lim_{t\to\infty} S_t f(z) &= \int_M f\phi \,\mathrm{dvol}_{g} \\ &= \int_M f\phi_{\mathrm{b}} \,\mathrm{dvol}_{g}.\end{aligned}$$

Thus

$$0 = \int_M f[\phi'_{\mathrm{b}'}\phi_{\mathrm{b}} - \phi_{\mathrm{b}}] \,\mathrm{dvol}_g \text{ for all basic } f,$$

hence

$$\phi'_{\mathrm{b}'} \equiv 1, \tag{48}$$

since ϕ_{b} never vanishes [AL, Prop. 2.2].

Remark 1. The above argument shows that for any smooth basic function $\psi > 0$ on M, there exists a bundle-like metric g', obtained from g by a suitable $\mathcal{F}$-dilation, such that $\psi = \phi'_{\mathrm{b}'}$.

We recall that the exterior derivative d preserves the basic functions (and forms) $\mathcal{A}_{\mathrm{b}}$. Therefore, its adjoint δ preserves the L^2-orthogonal complement $\mathcal{A}_{\mathrm{b}}^{\perp}$. By Cor. 5.3, A preserves the basic functions C_{b}, hence its adjoint A^* leaves $C_{\mathrm{b}}^{\perp}$ invariant. Writing $\phi = \phi_{\mathrm{b}} + \phi_{\mathrm{o}}$ as the sum of its basic and orthogonal components, and using the fact that ϕ_{b} and ϕ_{o} are smooth, we see that $A^*\phi_{\mathrm{o}} \in C_{\mathrm{b}}^{\perp}$. Since $A^* f = -\delta(df - f\kappa)/2$ by (44), we obtain $\delta(d\phi_{\mathrm{o}} - \phi_{\mathrm{o}}\kappa) \in C_{\mathrm{b}}^{\perp}$. Together with the argument leading to (48), this implies:

Theorem 6.2. *Let a bundle-like metric g be given. Then there exists another bundle-like metric g' on M, obtained by a dilation of g as in Eq. (46), with the property that κ_b is basic-harmonic, i.e., $\delta_b \kappa_b = 0 = d\kappa_b$.*

Proof. By definition, $\delta_{\mathrm{b}} = P_{\mathrm{b}} \circ \delta$, where P_{b} is the L^2 projection onto the basic complex. According to [AL, Cor. 3.5], $d\kappa_{\mathrm{b}} = 0$. On the other hand, using $A^*\phi = 0$ and $\phi = \phi_{\mathrm{b}} + \phi_{\mathrm{o}}$, we have

$$\delta(d\phi_{\mathrm{b}} - \phi_{\mathrm{b}}\kappa) = -\delta(d\phi_{\mathrm{o}} - \phi_{\mathrm{o}}\kappa) \in C_{\mathrm{b}}^{\perp}.$$

Clearly, $\phi_{\mathrm{b}}\kappa_{\mathrm{o}} \in \mathcal{A}_{\mathrm{b}}^{\perp}$, so $\delta(\phi_{\mathrm{b}}\kappa_{\mathrm{o}}) \in C_{\mathrm{b}}^{\perp}$ and therefore

$$\delta(d\phi_{\mathrm{b}} - \phi_{\mathrm{b}}\kappa_{\mathrm{b}}) \in C_{\mathrm{b}}^{\perp}. \tag{49}$$

Using the metric g', we may suppose that ϕ_{b} is identically equal to 1. Then $\delta\kappa_{\mathrm{b}} \in C_{\mathrm{b}}^{\perp}$, i.e., $\delta_{\mathrm{b}}\kappa_{\mathrm{b}} = 0$. □

Remark 2. This result is trivial if all basic functions are locally constant, because any divergence automatically integrates to zero. In the contrary case, however, dim $dC_{\mathrm{b}} = \infty$ and Theorem 6.2 solves an infinite-dimensional, global, nonlinear problem.

Remark 3. It is clear from Proposition 6.1 that $\phi = \text{ const } \iff \delta\kappa = 0$. Moreover, $\phi_{\mathrm{b}} = \text{ const } \iff \delta_{\mathrm{b}}\kappa = 0$. The implication $\Rightarrow$ was shown in the proof of Theorem 6.2. Conversely, suppose that $\delta_{\mathrm{b}}\kappa = 0$. We always have $-\delta(d\phi_{\mathrm{b}} - \phi_{\mathrm{b}}\kappa_{\mathrm{b}}) \in C_{\mathrm{b}}^{\perp}(M)$, but this is equal to

$$\begin{aligned}
&\Delta\phi_{\mathrm{b}} + \phi_{\mathrm{b}}\delta\kappa_{\mathrm{b}} - \kappa_{\mathrm{b}}(\phi_{\mathrm{b}}) \\
&\quad = (2A\phi_{\mathrm{b}} - \kappa(\phi_{\mathrm{b}})) + \phi_{\mathrm{b}}\delta\kappa_{\mathrm{b}} - \kappa_{\mathrm{b}}(\phi_{\mathrm{b}}) \\
&\quad = 2A\phi_{\mathrm{b}} - 2\kappa_{\mathrm{b}}(\phi_{\mathrm{b}}) - \kappa_{\mathrm{o}}(\phi_{\mathrm{b}}) + \phi_{\mathrm{b}}\delta\kappa_{\mathrm{b}}.
\end{aligned}$$

The first two terms in the last line are in $C_{\mathrm{b}}(M)$, and by hypothesis the last term is in $C_{\mathrm{b}}^{\perp}(M)$. Moreover, $P_{\mathrm{b}}\,\kappa_{\mathrm{o}}(\phi_{\mathrm{b}}) = 0$, since $C_{\mathrm{b}}^{\perp} \ni \delta(\phi_{\mathrm{b}}\kappa_{\mathrm{o}}) = \phi_{\mathrm{b}}\delta\kappa_{\mathrm{o}} - \kappa_{\mathrm{o}}(\phi_{\mathrm{b}})$ gives $P_{\mathrm{b}}\kappa_{\mathrm{o}}(\phi_{\mathrm{b}}) = P_{\mathrm{b}}(\phi_{\mathrm{b}}\delta\kappa_{\mathrm{o}}) = \phi_{\mathrm{b}}\,P_{\mathrm{b}}\delta\kappa_{\mathrm{o}} = 0$. It follows that $(A - \kappa_{\mathrm{b}})\,\phi_{\mathrm{b}} = 0$, hence by the maximum principle for elliptic operators, $\phi_{\mathrm{b}} = \text{ const}$.

Although the content of Theorem 6.2 is in no way changed, it takes a somewhat nicer form (κ_{b} can be replaced by κ) if we assume the truth of a long-standing conjecture asserting the existence of a bundle-like metric with basic mean curvature. This conjecture has recently been proved by Domínguez.

Corollary 6.3. *Let M be a compact manifold equipped with a Riemannian foliation, and let g be a bundle-like metric for which κ is basic [Dom]. Then g can be*

dilated to obtain another bundle-like metric g' for which the mean curvature κ' is basic-harmonic.

Proof. If f is any smooth strictly positive function on M, its basic component is again smooth and strictly positive: $f_{\mathrm{b}} > 0$ ([AL, Prop. 2.2]). Thus we need only dilate g by ϕ_{b}; we saw in (48) that ϕ' for the new metric g' has constant basic part. Since $\kappa' = \kappa - d_{1,0} \log \phi_{\mathrm{b}} = \kappa - d \log \phi_{\mathrm{b}}$ is again basic, the result follows from the primed analog of (49), in which all quantities are for the metric g'. $\square$

The above corollary fits well with the Hodge decomposition for the basic complex (see, e.g., [KT]). This gives an orthogonal decomposition

$$\mathcal{A}_{\mathrm{b}}(M) = \text{ im } d_{\mathrm{b}} \oplus H_{\mathrm{b}} \oplus \text{ im } \delta_{\mathrm{b}},$$

where d_{b} is d restricted to the basic forms and $\delta_{\mathrm{b}} = P_{\mathrm{b}} \circ \delta$, with P_{b} the L^2 projection onto the basic complex. The space H_{b} consists of those forms α satisfying $d_{\mathrm{b}}\alpha = 0 = \delta_{\mathrm{b}}\alpha$ and is finite-dimensional. Since κ basic is equivalent to $d\kappa = 0$, we know *a priori* only that $\kappa \in \text{im } d_{\mathrm{b}} \oplus H_{\mathrm{b}}$. The Corollary asserts that we can arrange for κ to lie in the finite-dimensional space H_{b}. This result does not seem to follow from the Hodge decomposition. For suppose that a bundle-like metric g with κ basic has been found. Then $d\kappa = 0$ and we can write $\kappa = d_{\mathrm{b}} f + h$, where f is basic and h is basic-harmonic. A natural thing to try is to set $\lambda = e^f$ and dilate g by λ to get $\kappa' = \kappa - d_{1,0} f = h$. Then κ' is again basic, but h is in general not basic-harmonic for the new metric g'. More precisely, by Remark 3 and the argument leading to (48), $h = \kappa'$ is basic-harmonic for $g' \iff \phi'_{\mathrm{b}'} = e^{-f} \phi_{\mathrm{b}}$ is constant $\iff \kappa = d_{\mathrm{b}} \log \phi_{\mathrm{b}} + h$.

7. AN EXAMPLE

We conclude with an example [Car]. Consider the manifold $M' = T \times \mathbb{R}$ where T is the 2-torus, and let $A \in SL(2, \mathbb{Z})$ have trace > 2. Then A has distinct real (irrational) eigenvalues λ and $1/\lambda$ with associated eigenvectors V_1 and V_2. It defines an orientation-preserving diffeomorphism of $T = \mathbb{R}^2/\mathbb{Z}^2$. The direction determined by V_1, say, defines a flow on M by

$$\psi_s((x, y), t) = ((x, y) + sV_1, t)$$

for $s \in \mathbb{R}$. The integers $\mathbb{Z}$ act on M' by $((x, y), t)^m = (A^m((x, y)), t + m)$, (x, y) a general point in T. Because V_1 is an eigenvector of A, the flow defined by ψ induces a one-dimensional Riemannian foliation $\mathcal{F}$ on the compact quotient

manifold $M = M'/\mathbb{Z}$. The nonconstant function $F([(x,y),t]) = \sin(2\pi t)$ is well-defined on M and is basic, hence the space $d_{\mathrm{b}}(C_{\mathrm{b}}(M))$ is infinite-dimensional. Carrière shows that $(M,\mathcal{F})$ admits a transverse Lie structure modeled on the affine group $\mathbb{R}^2$. This feature enabled him to prove directly that the second basic cohomology group vanishes: $H^2_{\mathrm{b}} = 0$. It follows that there exists no bundle-like metric for which $\kappa = 0$. For more details, we refer to Chapter 10 of [T]. Since κ is nontrivial (in a rather strong sense) and nonconstant basic functions exist, Theorem 6.2 has content in this case.

Let us examine in more detail what our results say in the context of the above example. We take the leaf coordinate x to be along V_1 and the transverse coordinates y and t to be along V_2 and the t axis, respectively. The local model space $\mathbb{R}^2$ is identified with the affine group $GA(2)$ with group law $(y,t)\circ(y',t') = (\lambda^{-t}y' + y, t+t')$. The transverse metric g_T is taken to be any left-invariant metric on $GA(2)$. This amounts to assigning a metric arbitrarily at the identity element $(0,0)$ and transporting it by left multiplication. Thus,

$$\begin{aligned} g_T\Big|_{(y,t)}(\lambda^{-t}\frac{\partial}{\partial y}, \lambda^{-t}\frac{\partial}{\partial y}) &= g_T\Big|_{(0,0)}(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}), \\ g_T\Big|_{(y,t)}(\lambda^{-t}\frac{\partial}{\partial y}, \frac{\partial}{\partial t}) &= g_T\Big|_{(0,0)}(\frac{\partial}{\partial y}, \frac{\partial}{\partial t}), \\ g_T\Big|_{(y,t)}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= g_T\Big|_{(0,0)}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}). \end{aligned}$$

In particular, there is no need to take $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial t}$ to be orthonormal at $(0,0)$, though of course we could. By construction, the metric g_T is invariant under the identification $(x,y,0) = (\lambda x, \lambda^{-1}y, 1) \in T\times\mathbb{R}$ in the definition of M.

As stated after Eq. (3), given any Riemannian metric g' on M, we obtain a bundle-like metric compatible with g_T by setting $g(X,Y) = g'(PX,PY) + g_T(\overline{X},\overline{Y})$. We could take g' to come from the standard metric g'' on $T\times\mathbb{R}$, except within a buffer layer $T\times[1-c,1)$, where g'' must be deformed so as to be consistent with the identification $(x,y,0)\sim(A(x,y),1)$ and give a well-defined metric g' on the quotient M. Many other choices of g' and hence g are possible; for instance, $T = S^1\times S^1$ and we could perturb the metrics on each of the circle factors. With the standard choice, $\frac{\partial}{\partial y}$ will not be orthogonal to $\frac{\partial}{\partial x}$.

To find the mean curvature κ in local coordinates we use the Koszul formula, which requires computing the Lie brackets $[e_1,e_2]$ and $[e_1,e_3]$ for an orthonormal frame $\{e_1,e_2,e_3\}$ with e_1 proportional to $\frac{\partial}{\partial x}$ and e_2 and e_3 linear combinations

of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, and $\frac{\partial}{\partial t}$, all coefficients depending on the metric g. This can be done explicitly but is not particularly illuminating. Furthermore, there is little hope of actually finding the function $\phi_{\mathrm{b}}(t)$ explicitly.

We now consider what the Corollary of Theorem 6.2 says in the present situation. Since λ is irrational, for each $t \in [0, 1)$ every leaf meeting the torus $T \times \{t\}$ is dense in it, hence the basic functions F on M depend only on the t coordinate and can be identified with the smooth functions on $\mathbb{R}^1$ with period 1. By [Dom, Theorem 4.18], given any g_T there exists a bundle-like metric g satisfying (3) for which κ is basic. Dilating by ϕ_{b}, we can achieve in addition that $\delta_{\mathrm{b}}\kappa = 0$, i.e., $\int_M F'(t)(dt, \kappa)\mathrm{dvol}_g = 0$ for every smooth function F with period 1 in t. We set $h(t) = (dt, \kappa)$, which is a basic function because κ is basic and g is bundle-like. Taking $F(t)$ to be $\sin(2\pi mt)$ or $\cos(2\pi mt)$ for $m \in \mathbb{Z}$, it follows that $\int_M \cos(2\pi mt)h(t)\mathrm{dvol}_g = 0$ and $\int_M \sin(2\pi mt)h(t)\mathrm{dvol}_g = 0$ for all m, except that $m = 0$ must be excluded in the first case. Letting F be any smooth periodic function with period 1 and expanding F in a Fourier series, we conclude that

$$\int_M F(t)h(t)\mathrm{dvol}_g = CF_0, \tag{50}$$

where $C = \int_M h(t)\mathrm{dvol}_g$ and $F_0 = \int_0^1 F(t)dt$. This equality extends by continuity to periodic F in $L^1[0, 1]$.

Replacing dt by $-dt$ if necessary, we may suppose that $C \geqslant 0$. If $C = 0$ then (50) with $F = h$ shows that $h \equiv 0$, so let us take $C \neq 0$. Taking F to be the characteristic function of $[\alpha, \beta]$, we find that $\int_{\alpha \leqslant t \leqslant \beta} h(t)\mathrm{dvol}_g = C(\beta - \alpha)$ for all $\alpha, \beta \in [0, 1]$. It follows that

$$h(t)/C = \frac{d\mu_L}{d\mu}(t), \tag{51}$$

the Radon–Nikodym derivative of Lebesgue measure on $[0, 1]$ with respect to the measure μ defined on $[0, 1]$ by $\mu[\alpha, \beta] = \int_M \chi_{\{\alpha \leqslant t \leqslant \beta\}}(x, y, t)\ \mathrm{dvol}_g$. Thus Cor. 6.3 is equivalent to the assertion that $(dt, \kappa) = \int_M (dt, \kappa)\mathrm{dvol}_g\ d\mu_L/d\mu$.

We observe parenthetically that unless $h \equiv 0$, we must have $h(t) > 0$ for all t, since (50) and the monotone convergence theorem imply that $\mathrm{Vol}(M) = C \int_0^1 \frac{1}{h(t)}\, dt$. Since h is smooth, if it ever vanished then the integral could not converge. In particular, if (dt, κ) ever vanishes (e.g., if κ vanishes at some point), then it vanishes identically. We recall here Carrière's result that there exists no bundle-like metric for which $\kappa \equiv 0$.

Passing to the general case, we expect Theorem 6.2 to be nontrivial for Riemannian foliations of higher codimension. Provided the maximum dimension of

the leaf closures is strictly less than the dimension of M, one expects nonconstant basic functions to exist.

8. CONCLUDING REMARKS

Examination of the proof of Theorem 6.2 suggests that it might be possible to construct a proof that avoids probability theory. Indeed, neither the ergodic theorem nor the existence of ϕ requires probability; moreover, Cor. 5.3 can be established independently (it can be deduced, for instance, from Proposition 4.1 in [PR]). Furthermore, as noted after Eq. (47), no appeal to Lemma 3.5 is necessary. However, the proof of (47), which is based on Lemma 5.2, does require Lemma 3.4 (and also the reduction to ${}^{\mathcal{F}}\mathcal{O}(M)$). Thus, as far as Theorem 6.2 is concerned, the role of the probability theory is confined to the proof that S_t preserves the basic functions. But this property is much stronger than Cor. 5.3. Indeed, according to the theory of semigroups (see, e.g., [Y, Chap. IX]), it amounts to the following: For each $f \in C^2(M)$ and $\alpha \geqslant 0$, if $(1 - \alpha A)f$ is basic then f is basic. I do not see how to prove this without using Lemma 3.4.

Finally, it may be worth noting that there is a suggestive analogy between ϕ and the function λ considered in [Dom], which satisfies $d_{1,0}\lambda - \lambda\kappa_{\mathrm{o}} \in \delta_{\mathcal{F}}\mathcal{A}^{1,1}$. Domínguez's proof might be greatly simplified, and its geometric content made more apparent, if λ could be replaced by ϕ. This was actually one of the original motivations for the present work. One can show that $d_{1,0}\lambda - \lambda\kappa_{\mathrm{o}} \in \overline{\delta_{\mathcal{F}}\mathcal{A}^{1,1}}$, the Fréchet closure of the image $\delta_{\mathcal{F}}\mathcal{A}^{1,1}$, if and only if $\lambda_{\mathrm{b}} = \text{const}$. Hence if ϕ can replace λ then we must have $\phi_{\mathrm{b}} = \text{const}$; that this can be achieved is the content of Theorem 6.2. But we are unable to proceed further using our methods, because they give no control over the basic-orthogonal part ϕ_{o}.

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2316 Wright Circle, Round Rock, TX 78664
E-mail address: Alanndg@aol.com